ORIGINAL PAPER

CC BY 4.0

DOI: 10.26794/2587-5671-2020-24-6-92-107
UDC 336.763(045)
JEL G11, G12, G17, G32

New Risk Measures "*VaR* to the Power of *t*" and "*ES* to the Power of *t*" and Distortion Risk Measures

V.B. Minasyan
Higher School of Finance and Management, Russian Presidential Academy of National Economy and Public Administration, Moscow, Russia
https://orcid.org/0000-0001-6393-145X

ABSTRACT

Distortion risk measures have been popular in financial and insurance applications in recent years due to their attractive properties. **The aim** of the article is to investigate whether risk measures "*VaR* in the power of *t*", introduced by the author, belong to the class of distortion risk measures, as well as to describe the corresponding distortion functions. The author introduces a new class of risk measures "*ES* to the power of *t*" and investigates whether it belongs to distortion risk measures, and also describes the corresponding distortion functions. The author used the composite **method** to design new distortion functions and corresponding distortion risk measures, to prove that risk measures "*VaR* to the power of *t*" and "*ES* to the power of *t*" belong to the class of distortion risk measures. The paper presents examples to illustrate the relevant concepts and results that show the importance of risk measures "*VaR* to the power of *t*" and "*ES* to the power of *t*" as subsets of distortion risk measures that allow identifying various financial catastrophic risks. The author **concludes** that risk measures "*VaR* to the power of *t*" and "*ES* to the power of *t*" can be used in risk management of companies when assessing remote, highly catastrophic risks.

Keywords: catastrophic risks; distortion risk measures; distortion functions; composite method; coherent risk measures; risk measures "*VaR* to the power of *t*"; risk measures "*ES* to the power of *t*"

For citation: **Minasyan V.B. New risk measures "*VaR* to the power of *t*" and "*ES* to the power of *t*" and distortion risk measures. *Finance: Theory and Practice.* 2020;24(6):92-107. (In Russ.). DOI: 10.26794/2587-5671-2020-24-6-92-107**

© Minasyan V.B., 2020

INTRODUCTION

The risk measure is a mapping ρ of a set of random variables X associated with risk portfolios of assets and/or liabilities (the result variables of these portfolios) into a real line R. In the following discussion, X will be represented as the value of the corresponding losses, i.e. positive values of the X variables will represent losses, while negative values will represent gains. Distortion risk measures represent a special and important group of risk measures widely used in finance and insurance as a calculation of capital requirements and the principles of calculating indicators related to risk appetite for the regulator and company executive. Several popular risk measures have proven to belong to distortion risk measures. For example, value at risk (VaR), tail value at risk, expected shortfall (ES) [1–3] or Wang's distortion measure [4]. Distortion risk measures satisfy the most important properties that a "good" risk measure should have, including positive homogeneity, translation invariance, and monotonicity [5].

In our previous works, we introduced a family of risk measures called "VaR to the power of t" ($VaR_p^{(t)}[X]$) for any confidence probability p and any real $t \geq 1$ [6–8]. In these works, computational formulas for risk measures $VaR_p^{(t)}[X]$ were obtained. Also, the relationships between these measures and the measures such as $ES_p[X]$ for some specific loss distribution laws were investigated. We revealed that the relative persistence level for each measure can depend both on the loss distribution law and on the confidence probability with which these measures are calculated. However, for almost all loss distribution laws and for all confidence probabilities of practical interest, risk measure $VaR_p^{(t)}[X]$ for any real $t \geq 2$ turns out to be more persistent, providing a more "careful" risk assessment than, say, risk measure $ES_p[X]$.

D. Denneberg, S. Wang and J. Dhaene [9, 10] proved that when the corresponding distortion function is concave, the distortion risk measure is also sub-additive. VaR is one of the most popular risk measures used in risk management and banking supervision because of its computational simplicity and for some regulatory reasons, regardless of its shortcomings as a risk measure. For example, VaR is not a sub-additive risk measure [11, 12]. Being coherent [2, 3], ES risk measure is only interested in losses in excess of VaR and ignores useful information about the distribution of losses below Va R.

L. Zhu and H. Li [13] presented and studied the tail distortion risk measure reformulated by F. Yang [14] as follows. For the distortion function g, the tail distortion risk measure at the confidence level p for the loss variable X is defined as the distortion risk measure with the distortion function:

$$g_p(x) = \begin{cases} g(\frac{x}{1-p}), if \ 0 \leq x \leq 1-p \\ 1, if \ 1-p < x \leq 1. \end{cases}$$

C. Yin and D. Zhu [15] described three methods for building distortion risk measures: the composite method, the mixing method, and the copula-based approach. We will use the results of this work.

Many researchers have proposed new classes of distortion measures.

For example, J. Belles-Sampera, M. Guillén, M. Santolino [16] proposed a new class of distortion risk measures called GlueVaR risk measures to extend VaR and ES. They can be expressed as a combination of VaR and ES indicators at different levels of confidence probabilities. They obtained closed-form analytical expressions for the most commonly used distribution functions in finance and insurance. The subfamily of these risk measures satisfies the tail sub-additivity property, which means that diversification benefits can persist, at least in certain cases. The application of GlueVaR risk measures related to capital allocation was discussed by J. Belles-Sampera, M. Guillén, M. Santolino [17].

U. Cherubini and S. Mulinacci [18] proposed a class of distortion measures based on contamination from an external "scenario" variable.

For a scenario-dependent variable whose risk is modeled by a copula function with horizontally concave portions, they give conditions for the coherence axiom and offer examples of this class measures based on the copula function.

It would be interesting to investigate the relationship between two classes of risk measures: distortion risk measures and *VaR* to the power of *t*.

We introduce a family of new risk measures "*ES* to the power of *t*" ($ES_p^{(t)}[X]$) at any confidence probability *p* and any real $t \geq 1$. We investigated the relationship of two classes of risk measures: distortion and *ES* to the power of *t*. It is proved that risk measures "*ES* to the power of *t*" is a subset of distortion risk measures. Thus, for any $t \geq 1$, any risk measure "*ES* to the power of *t*" is a distortion risk measure with a certain distortion function. In this case, this distortion function will be presented.

It is hard to believe in a unique risk measure that can encompass all of its characteristics. It does not exist. Moreover, since a single number is associated with each risk measure, it cannot exhaust all information about the risk. According to [8], risk measures "*VaR* to the power of *t*" allow by changing the value of *t*, to study the right tail of the loss distribution with any accuracy required for the given case, i.e. investigate the tail of the distribution as carefully as it is necessary in the given circumstances. It is prudent to look for risk measures ideal for a particular problem. Since all the proposed risk measures are flawed and limited in application, selecting the appropriate risk measure is still relevant in risk management.

DISTORTION RISK MEASURES

Distortion functions

A distortion function g: [0, 1] → [0, 1] is a non-decreasing function such that g $g(0) = 0, g(1) = 1$. Many distortion functions *g* have already been proposed in the literature. Some commonly used distortion functions are listed here.

The work by M. Denuit, J. Dhaene, M. Goovaerts and R. Kaas [12] presents the summary of other distortion functions.

- Function $g(x) = 1_{\{x>1-p\}}$, where 1_A is the indicator function and equals 1 at event A, and equals 0 otherwise, is a concave distortion function. Here, in applications, *p* will represent the preselected confidence level with which the corresponding risk measure is intended to be calculated.
- Incomplete beta-function

$$g(x) = \frac{1}{\beta(a,b)} \int_0^x t^{a-1}(1-t)^{b-1} dt ,$$

where *a* > 0 and *b* > 0 are the parameters and

$$\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt.$$

In particular, if *b* = 1, we obtain the power distortion function $g(x) = x^a$, and if *a* = 1, we obtain the dual-power distortion function $g(x) = 1-(1-x)^b$.

- Power distortion $g(x) = x^\alpha$ is a concave distortion function if $0 < \alpha < 1$ and a convex distortion function if $\alpha > 1$.
- Exponential distortion $g(x) = \frac{e^x - 1}{e - 1}$ is a convex distortion function.
- Sinusoidal distortion $g(x) = \sin\frac{\pi}{2}x$ is a concave distortion function.
- Function $g(x) = xe^{1-x}$ is a concave distortion function.
- Logarithmic distortion $g(x) = \frac{\ln(x+1)}{\ln 2}$ is a concave distortion function.
- Distortion Wang $g(x) = \Phi(\Phi^{-1}(x) + \Phi^{-1}(p))$, $0 < p < 1$, where Φ is the standard normal distribution function. Obviously, this is an increasing function [since these are functions $\Phi(x)$ and $\Phi^{-1}(x)$] and

$g(0) = \Phi(\Phi^{-1}(0) + \Phi^{-1}(p)) = \Phi(-\infty) = 0$ and

$g(1) = \Phi(\Phi^{-1}(1) + \Phi^{-1}(p)) = \Phi(+\infty) = 1$, and

$$g(\frac{1}{2}) = \Phi(\Phi^{-1}(\frac{1}{2}) + \Phi^{-1}(p)) = \Phi(\Phi^{-1}(p)) = p$$

- Lookback distortion

$g(x) = x^p(1 - p\ln x), p \in (0,1]$.

Obviously, this is an increasing function, which is easy to check: $g'(x) = -p^2 x^{p-1} \ln x > 0$ if $x \in [0,1]$, and $g(0) = \lim_{x \to +0} g(x) = 0$ and $g(1) = 1$.

- Identity function $g(x)=x$ is the smallest concave distortion function and also the largest convex distortion function.
- $g_0(x)=1_{\{x>0\}}$ is concave on [0,1] and is the largest of all non-identical concave distortion functions. $g^0(x)=1_{\{x=1\}}$ is convex on [0,1] and is the smallest of all non-identical convex distortion functions.
- For 0 < p <1, $g(x)=\min\{\frac{x}{1-p},1\}$ is the smallest concave distortion function of all $g(x)\geq 1_{\{x>1-p\}}$.

Distortion risk measures

If (Ω, F, P) is a common probability space where all random variables that represent the risks are defined. If F_X is an integral distribution function of random variable *X*, we denote the dual distribution function as $\bar{F}_X$, i.e. $\bar{F}_X(x)=1-F_X(x)=P\{X>x\}$.

Let *g* be the distortion function.

Distorted expectation of random variable X is $\rho_g[X]$ and is defined as

$$\rho_g[X]=\int_0^{+\infty} g(\bar{F}_X(x))dx+\int_{-\infty}^{0}[g(\bar{F}_X(x))-1]dx, \qquad (1)$$

provided that at least one of the two integrals above is finite. If *X* is a non-negative random variable, then ρ_g is simplified to

$$\rho_g[X]=\int_0^{+\infty} g(\bar{F}_X(x))dx.$$

This definition implies that if the distortion function is an identical function, i.e. $g(x)=x$, then the distorted expectation coincides with the usual expectation: $\rho_g[X]=E[X]$.

Due to the fact that the expected value of a random variable is considered the most important way to assess the future value of random variable *X*, we assume that since risks arise due to some value deviation of a random variable from its expected value, then risk measures can be modeled as a "distortion" of the expected value using the appropriate distortion function.

Distorted expectation $\rho_g[X]$ is called the *distortion risk measure with distortion function g* [19].

We can prove that, as was first observed by M. Denuit, J. Dhaene, M. Goovaerts and R. Kaas [12], the known risk measure VaR [1–3] is a distorted risk measure corresponding to distortion function $g(x)=1_{\{x>1-p\}}$, $p\in(0,1)$, i.e. the following definition is true.

Definition 1 [19]

For the distortion function $g(x)=1_{\{x>1-p\}}$, $p\in(0,1)$, if distribution function F_X is continuous, the corresponding risk measure is $\rho_g[X]=VaR_p[X]$.

J. Dhaene et al. [19] also proved two important facts that describ the relationship of all distortion risk measures obtained by distortion functions that are continuous on the right on [0,1) or left on (0,1] with the risk measures Va R.

Theorem 1

When *g* is a continuous distortion function on the right on [0,1), the distorted expectation $\rho_g[X]$ has the following representation:

$$\rho_g[X]=\int_{[0,1]} VaR^+_{1-q}[X]dg(q),$$

where $\mathrm{VaR}^+_p[X]=\sup\{x\,|\,F_X(x)\leq p\}$

Theorem 2

When *g* is a continuous distortion function on the left on [0,1), the distorted expectation $\rho_g[X]$ has the following representation:

$$\rho_g[X]=\int_{[0,1]} VaR_{1-q}[X]dg(q)=\int_{[0,1]} VaR_q[X]d\bar{g}(q),$$

Where $\mathrm{VaR}_p[X]=\inf\{x\,|\,F_X(x)\geq p\}$ and $\bar{g}(q)=1-g(1-q)$ — are the dual distortion to *g*.

Obviously, $\bar{\bar{g}}=g$, and *g* is continuous on the left if and only if $\bar{g}$ is continuous on the right; *g* is concave if and only if $\bar{g}$ is convex.

Distortion risk measures are a special class of risk measures introduced by D. Denneberg [9] and modified by S. S. Wang [4, 20].

Distortion risk measures satisfy many properties, including positive homogeneity, translation invariance, and monotonicity. A risk measure is coherent if it satisfies the following set of four properties [11, 21]:

(M) monotonicity: $\rho(X) \leq \rho(Y)$ if $P(X \leq Y) = 1$;

(*P*) positive homogeneity: for any positive constant $c > 0$ and loss X, $\rho(cX) = c\rho(X)$;

(*S*) sub-additivity: for any losses X, Y, then $\rho(X+Y) \leq \rho(X)+\rho(Y)$;

(T) translation invariance: if c is constant, then $\rho(X+c) = \rho(X) - c$.

Risk measure ρ is called a convex risk measure if it satisfies the properties of monotonicity, translation invariance and the following convexity property:

(*C*) convexity: $\rho(\lambda X + (1-\lambda) Y) \leqslant \lambda \rho(X) + (1-\lambda)\rho(Y)$, $0 \leqslant \lambda \leqslant 1$.

Obviously, under the assumption of positive homogeneity, monotonicity, and translation invariance, the convexity of the risk measure is equivalent to sub-additivity.

Another distortion risk measure [19], besides VaR, is the well-known measure ES (expected shortfall), conditional VaR [1–3].

Definition 2 [19]

For distortion function $g(x) = \min\{\frac{x}{1-p}, 1\}$, $p \in [0,1]$ under the assumption that the distribution function is continuous, the corresponding distorted risk measure is $\rho_g[X] = ES_p[X]$.

The following theorem [17] is useful and can be used to order distortion risk measures in terms of their distortion functions.

Theorem 3 [17]

If $g(x) \leq g^*(x)$ for $x \in [0,1]$, then $\rho_g[X] \leqslant \rho_{g^*}[X]$ for any random variable X.

DISTORTION RISK MEASURES *VAR* TO THE POWER OF *T*, *t* ⩾ 1, ($VaR_p^{(t)}$)

Today, risk measure VaR is probably the second most commonly used risk measure, both in theory and practice, after volatility (standard deviation). Since the end of the twentieth century, ES (Expected Shortfall) measure, conditional VaR, the measure of expected tail losses exceeding VaR, has found sufficient application in risk management. ES is perceived as a risk measure that specifies VaR measure, more conservative, considering tail losses, unlikely but large ("black swan").

The concept of a new measure "VaR squared" $VaR^{(2)}$ [6, 7] estimates risks more conservatively than VaR and is often more conservative than ES, assessing risk as a certain threshold value that is not overcome with a given probability (as VaR), and not as some average value from the set of "bad", tail loss values, like ES.

Following the ideas in [6, 7], in [8] we introduced the concept of risk measures VaR to any power $t \geq 1$, and derived formulas that allow to calculate $VaR^{(t)}$ as usual measure VaR with a certain modified confidence probability.

The concept of *VaR* to any natural power $VaR^{(n)}$ [6, 7] introduced a new risk measure to supplement VaR, tracking rare tail events associated with great financial losses.

Risk measure VaR squared" $VaR^{(2)}$ *with a confidence level p* is the value that will not be exceeded by the loss if its threshold value VaR_p is exceeded with a confidence level p during a given time.

Work [8] presents the following formula:

$$VaR_p^{(2)}[X] = VaR_{1-(1-p)^2}[X]. \tag{2}$$

Thus, to calculate new catastrophic risk measure "VaR squared", a general formula has been obtained. We should just calculate risk measure VaR with confidence level $1-(1-p)^2$.

The concept of $VaR^{(2)}$ in [8] was generalized considering the fact that the confidence probability p' when determining $VaR^{(2)}$, i.e. the threshold value that the profit will not exceed (the loss will exceed), provided that it is not exceeded (exceeded) by VaR_p with probability p', may differ from p. This risk measure, which can be called "bi-VaR", was designated as $VaR_{p,p'}^{(2)}$ and the following formula was obtained:

$$VaR_{p,p'}^{(2)}[X] = VaR_{1-(1-p)(1-p')}[X]. \tag{3}$$

We will introduce the concept of risk measures *VaR to the power of n*, where n is any natural number, and will give formulas to calculate risk measures *VaR to the power of n*, $VaR^{(n)}$ [8].

We represent usual risk measure VaR as:

$$VaR_p^{(1)}[X] = VaR_p[X] = VaR_{p_1}[X],$$

where $p_1 = 1-(1-p)$.

According to the formula,

$VaR_p^{(2)}[X] = VaR_{p_2}[X]$, where $p_2 = 1-(1-p_1)^2$.

Naturally, according to the definition, we can assume that "*VaR to the third power*" is just $VaR_{p_2,p}^{(2)}[X]$. Thus, we get that

$VaR_p^{(3)}[X] = VaR_{p_2,p}^{(2)}[X] = VaR_{p_3}[X]$, where according to (3) $p_3 = 1-(1-p_2)(1-p)$.

Following this way, we introduce risk measure "*VaR to the power of n*" for any natural number *n* as $VaR_{p_{n-1},p}^{(2)}[X]$, where $p_{n-1} = 1-(1-p)^{n-1}$ and we obtain that

$VaR_p^{(n)}[X] = VaR_{p_{n-1},p}^{(2)}[X] = VaR_{p_n}[X]$, where according to (3) $p_n = 1-(1-p_{n-1})(1-p)$.

The concept of risk measures "*VaR to the power of n*" was introduced [8] for any natural number *n* and the formula was obtained that reduces their calculations to the calculation of usual risk measure VaR with a confidence level changed in a certain way.

$$VaR_p^{(n)}[X] = VaR_{1-(1-p)^n}[X]. \quad (4)$$

Thus, to calculate risk measure $VaR_p^{(n)}$, we should just to calculate risk measure VaR with confidence level $1-(1-p)^n$.

RISK MEASURES "POLY-VaR"

We will introduce (like we did in [8]) a family of measures that generalize measures $VaR_p^{(n)}[X]$ and allow the confidence probabilities used for various powers of VaR to be different.

We will represent usual risk measure VaR as:

$$VaR_p[X] = VaR_{\tilde{p}_1}, \text{ where } \tilde{p}_1 = p_1 = p = 1-(1-p).$$

By formula (7), we introduce the concept of risk measure "poly-VaR to the second power", "bi-VaR":

$$VaR_{p_1,p_2}^{(2)}[X] = VaR_{\tilde{p}_2}[X], \text{ where } \tilde{p}_2 = 1-(1-p_1)(1-p_2).$$

Accordingly, risk measure "poly-VaR to the third power" is as follows:

$$VaR_{p_1,p_2,p_3}^{(3)}[X] = VaR_{\tilde{p}_2,p_3}^{(2)}[X] = VaR_{\tilde{p}_3}[X],$$

where $\tilde{p}_3 = 1-(1-\tilde{p}_2)(1-p_3)$.

Thus, risk measure "*poly-VaR to the power of n*" is defined as follows:

$$VaR_{p_1,p_2,\ldots,p_n}^{(n)}[X] = VaR_{\tilde{p}_{n-1},p_n}^{(2)}[X] = VaR_{\tilde{p}_n}[X],$$

where $\tilde{p}_n = 1-(1-\tilde{p}_{n-1})$.

Work [8] provides the following formula to calculate the *poly-VaR to the power of n*":

$$VaR_{p_1,p_2,\ldots,p_n}^{(n)}[X] = VaR_{1-(1-p_1)(1-p_2)\ldots(1-p_n)}[X], \quad (5)$$

that expresses it in terms of usual risk measure VaR with the confidence probability recalculated in a certain way.

RISK MEASURE VAR TO ANY REAL POWER $t \geqslant 1$, $VaR_p^{(t)}[X]$

Any real number $t \geq 1$ can be unambiguously represented as:

$t = k + \alpha$, where *k* is a natural number, and α is a real number, and $0 \leq \alpha < 1$. Obviously, *k* is the integer part of *t*, and α is its fractional part.

Naturally, risk measure *VaR* to any real power $t \geq 1$, $VaR_p^{(t)}[X]$ is as follows [8]:

$$VaR_p^{(t)}[X] = VaR^{(k+1)}_{\underbrace{p,p,\ldots,p}_{k},\alpha p}[X]. \quad (6)$$

In particular, using formulas (5) and (6), we have:

$$VaR_p^{(1+\alpha)}[X] = VaR_{p,\alpha p}^{(2)}[X] = VaR_{1-(1-p)(1-\alpha p)}[X] \quad (7)$$

and

$$VaR_p^{(2+\alpha)}[X] = VaR_{p,p,\alpha p}^{(3)}[X] = VaR_{1-(1-p)^2(1-\alpha p)}[X] \quad (8)$$

etc.,

$$VaR_p^{(t)}[X] = VaR_p^{(k+\alpha)}[X] = VaR_{1-(1-p)^k(1-\alpha p)}[X] \quad (9)$$

By means of risk measures $VaR_p^{(t)}[X]$, a risk manager can research the left tail of the profit distribution law for confidence probabilities that are multiples of the initial confidence probability p, as well as the fraction of this probability, to obtain very detailed information about less probable but more catastrophic risks.

NEW RISK MEASURES *ES* TO ANY POWER OF *T*, $t \geqslant 1$, $ES_p^{(t)}[X]$

We have already discussed an important risk measure, $ES_p[X]$ (Expected Shortfall) risk measure (conditional VaR), the measure of expected tail losses exceeding VaR. It is used as a risk measure, specifying VaR measure, more conservative, considering tail losses, unlikely, but great. In the second section, we described risk measure $VaR_p^{(t)}[X]$, which at $t \geq 2$ often gives a more conservative risk assessment than $ES_p[X]$.

In this paper, we introduce a new family of risk measures "ES to the power of t" for any $t \geq 1$.

First, we will introduce the concept of new risk measure — *"ES squared"*.

Risk measure *"ES squared"* denoted as $ES_p^{(2)}[X]$, is the value of the expected tail losses exceeding $VaR_p^{(2)}[X]$, i.e. by definition $ES_p^{(2)}[X] = E[X \mid X > VaR_p^{(2)}[X]$. (Symbol $E[X|A]$ denotes the conditional mathematical expectation of the random variable X if event A takes place).

Note that since $VaR_p^{(2)}[X] = VaR_{1-(1-p)^2}[X]$, the value of $ES_p^{(2)}[X]$ can be obtained by averaging the values of corresponding $\mathrm{VaR_q}[X]$ to variable q on segment $[1 - (1-p)^2, 1]$.

If the loss distribution continues, we obtain the following useful representation for $ES_p^{(2)}[X]$:

$$ES_p^{(2)}[X] = \frac{1}{(1-p)^2} \int\limits_{[1-(1-p)^2,1]} VaR_q[X]dq. \quad (10)$$

By analogy with ES squared, we introduce the concept of new risk measure *ES to the power of n*, where n is any natural number.

Risk measure "ES to the power of n", which we will designate as $ES_p^{(n)}[X]$, is the value of the expected tail losses exceeding $VaR_p^{(n)}[X]$, i.e. by definition $ES_p^{(n)}[X] = E[X \mid X > VaR_p^{(n)}[X]$.

Note that since $VaR_p^{(n)}[X] = VaR_{1-(1-p)^n}[X]$, the value of $ES_p^{(n)}[X]$ can be obtained by averaging the values of corresponding $\mathrm{VaR_q}[X]$ to variable q on segment $[1 - (1-p)^n, 1]$.

If the loss distribution continues, we obtain the following useful representation for $ES_p^{(n)}[X]$:

$$ES_p^{(n)}[X] = \frac{1}{(1-p)^n} \int\limits_{[1-(1-p)^n,1]} VaR_q[X]dq. \quad (11)$$

Note that a useful formula is obtained from formula (11), which allows expressing $ES_p^{(n)}[X]$ by usual risk measure ES with the confidence probability changed in a certain way:

$$ES_p^{(n)}[X] = ES_{1-(1-p)^n}[X]. \quad (12)$$

Now we will introduce new concept "*ES* to the power of t", where t is any real number, $t \geq 1$. We represent t as: $t = k + \alpha$, where k is a natural number, and a is a real number $0 < \alpha < 1$.

We will call *risk measure "ES to the power of t"*, denoted as $ES_p^{(t)}[X]$, the value of the expected tail losses exceeding $VaR_p^{(t)}[X]$, i.e. by definition $ES_p^{(t)}[X] = E[X \mid X > VaR_p^{(t)}[X]$.

Note that since $VaR_p^{(t)}[X] = VaR_{1-(1-p)^k(1-\alpha p)}[X]$, the value of $ES_p^{(t)}[X]$ can be obtained by averaging the values of corresponding $\mathrm{VaR_q}[X]$ to variable q on segment $[1 - (1-p)^k(1-\alpha p), 1]$.

If the loss distribution continues, we obtain the following useful representation for $ES_p^{(t)}[X]$:

$$ES_p^{(t)}[X] = \frac{1}{(1-p)^k(1-\alpha p)} \int\limits_{[1-(1-p)^k(1-\alpha p),1]} VaR_q[X]dq. \quad (13)$$

Note that a useful formula is obtained from formula (13), which allows expressing $ES_p^{(t)}[X]$ by usual risk measure ES with the confidence probability changed in a certain way

$$ES_p^{(t)}[X] = ES_{1-(1-p)^k(1-\alpha p)}[X]. \quad (14)$$

The following relations are valid between all the introduced risk measures:

$$VaR_p[X] \le ES_p[X],\ VaR_p^{(2)}[X] \le ES_p^{(2)}[X], \ldots,$$
$$VaR_p^{(n)}[X] \le ES_p^{(n)}[X], \ldots$$
$$VaR_p[X] \le VaR_p^{(2)}[X] \le \ldots \le VaR_p^{(n)}[X] \le \ldots$$
$$ES_p[X] \le ES_p^{(2)}[X] \le \ldots \le ES_p^{(n)}[X] \le \ldots$$

However, the ratio between risk measures $ES_p^{(n)}[X]$ and $VaR_p^{(n+1)}[X]$ may depend on the distribution law X and even on the confidence level p [7].

METHODS FOR CREATING NEW DISTORTION FUNCTIONS AND DISTORTION RISK MEASURES

Distortion functions can be viewed as a starting point for a family of distortion risk measures. Thus, building and selecting distortion functions play an important role in developing families of risk measures with different properties. C. Yin and D. Zhu [15] consider three methods: the composite method, mixing methods and copula-based method, which allow building new classes of distortion functions and measures using the available ones.

In this work, we will discuss only the composite method.

Composite method

The first approach to building distortion functions is the composite method that uses a composition of distortion functions.

If $h_1, h_2, \ldots$ are distortion functions, we will define $f_1(x) = h_1(x)$ and complex functions $f_n(x) = f_{n-1}(h_n(x))$, $n = 1, 2, \ldots$ It is easy to check that $f_n(x)$, $n = 1, 2, \ldots$ are also distortion functions. If $h_1, h_2, \ldots$ are concave distortion functions, then each $f_n(x)$ is concave, and they satisfy the conditions: $f_1 \le f_2 \le f_3 \le \ldots$ and the corresponding risk measures satisfy (by Theorem 3) $\rho_{f_1}[X] \le \rho_{f_2}[X] \le \rho_{f_3}[X] \le \ldots$

We will consider two distortion functions g_1 and g_2. If $g_2(x) = \begin{cases} \dfrac{x}{1-p}, if\ 0 \le x \le 1-p \\ 1, if\ \ 1-p < x \le 1, \end{cases}$,

then

$$g_p(x) = g_1(g_2(x)) = \begin{cases} g_1(\dfrac{x}{1-p}), if\ 0 \le x \le 1-p \\ 1, if\ \ 1-p < x \le 1, \end{cases}.$$

Corresponding risk measure $\rho_{g_p}[X]$ is a tail distortion risk measure first presented by L. Zhu and H. Li [13] and reformulated by F. Yang [14]. In particular, in the space of continuous random variable losses X

$$\rho_{g_p}[X] = \int_0^\infty g_p(1 - P(X \le x \mid X > VaR_p[X]))dx.$$

If $g_1(x) = x^r, 0 < r < 1$

and $g_2(x) = \begin{cases} \dfrac{x}{1-p}, if\ 0 \le x \le 1-p \\ 1, if\ \ 1-p < x \le 1, \end{cases}$,

then

$$g_{12}(x) = g_1(g_2(x)) = \begin{cases} (\dfrac{x}{1-p})^r, if\ 0 \le x \le 1-p \\ 1, if\ \ 1-p < x \le 1, \end{cases}$$

and

$$g_{21}(x) = g_2(g_1(x)) = \begin{cases} \dfrac{x^r}{1-p}, if\ 0 \le x \le (1-p)^{\frac{1}{r}} \\ 1, if\ \ (1-p)^{\frac{1}{r}} < x \le 1, \end{cases}$$

Obviously, $g_1 < g_{21}$ and $g_2 < g_{12}$, so by Theorem 3 $\rho_{g_1}[X] < \rho_{g_{21}}[X]$ and $\rho_{g_2}[X] < \rho_{g_{21}}[X]$.

In reality, it is sometimes necessary to distort the initial distribution more than once.

We will consider a few more examples of distortion functions obtained by the composite method as a composition of known distortion functions and will study the corresponding risk distortion measures.

Case 1

We will study exponential distortion function $g(x) = \dfrac{e^x - 1}{e - 1}$ is a convex distortion function and indicator concave distortion function $1_{\{x > 1-p\}}$.

It is easy to check that the composition of any distortion function *g*(*x*) (in particular, this one) with $1_{\{x>1-p\}}$ in the following order $g(1_{\{x>1-p\}}) = 1_{\{x>1-p\}}$, i.e. it does not create a new distortion function. If we change the order of creating the superposition, i.e. consider distortion function $1_{\{x>1-p\}}(g(x))$.

However, since $h(x) = 1_{\{x>1-p\}}(g(x)) = 1_{\{g(x)>1-p\}}(x)$ and inequality $\frac{e^x - 1}{e-1} > 1-p$ is equivalent to inequality $x > \ln(1+(e-1)(1-p))$, then

$$h(x) = 1_{\{x>1-p\}}(g(x)) = 1_{\{x>\ln(1+(e-1)(1-p))\}}(x) = \\ = 1_{\{x>1-[1-\ln(1+(e-1)(1-p))]\}}(x).$$

According to Definition 1, $\rho_h[X] = VaR_{1-\ln(1+(e-1)(1-p))}[X]$ is distortion risk measure corresponding to the given distortion function, i.e. known risk measure VaR with the confidence level changed in such a way. This risk measure grows very slowly with an increase in confidence.

For example, if the initial confidence level is $p = 0.95$, then $\rho_h[X] \approx VaR_{0.032}[X]$.

Case 2

We will look at logarithmic distortion function $g(x) = \frac{\ln(x+1)}{\ln 2}$, a concave distortion function, as well as at indicative concave distortion function $1_{\{x>1-p\}}$.

Let's consider a distortion function built with this superposition: $1_{\{x>1-p\}}(g(x))$.

However, since $h(x) = 1_{\{x>1-p\}}(g(x)) = 1_{\{g(x)>1-p\}}(x)$ and inequality $\frac{\ln(x+1)}{\ln 2} > 1-p$ is equivalent to inequality $x > 2^{1-p} - 1$, then

$$h(x) = 1_{\{x>1-p\}}(g(x)) = 1_{\{x>2^{1-p}-1\}}(x) = \\ = 1_{\{x>1-[1-(2^{1-p}-1)]\}}(x) = 1_{\{x>1-[2-2^{1-p}]\}}(x).$$

According to Definition 1, $\rho_h[X] = VaR_{2-2^{1-p}}[X]$ is distortion risk measure corresponding to the given distortion function, i.e. known risk measure VaR with the confidence level changed in such a way. This risk measure grows fast with increasing confidence probability.

For example, if the initial confidence level is $p = 0.95$, then $\rho_h[X] \approx VaR_{0.97}[X]$.

Case 3

We will look at sinusoidal distortion function $g(x) = \sin\frac{\pi}{2}x$, a concave distortion function, as well as at indicative concave distortion function $1_{\{x>1-p\}}$.

Let's consider a distortion function built with this superposition: $1_{\{x>1-p\}}(g(x))$.

However, since $h(x) = 1_{\{x>1-p\}}(g(x)) = 1_{\{g(x)>1-p\}}(x)$ and inequality $\sin\frac{\pi}{2}x > 1-p$ is equivalent to inequality $x > \frac{2}{\pi}\arcsin(1-p)$, then

$$h(x) = 1_{\{x>1-p\}}(g(x)) = 1_{\{x>\frac{2}{\pi}\arcsin(1-p)\}}(x) = \\ = 1_{\{x>1-[1-\frac{2}{\pi}\arcsin(1-p)]\}}(x).$$

According to Definition 1,

$\rho_h[X] = VaR_{1-\frac{2}{\pi}\arcsin(1-p)}[X]$ is distortion risk measure corresponding to the given distortion function, i.e. known risk measure VaR with the confidence level changed in such a way. This risk measure grows fast with increasing confidence probability.

For example, if the initial confidence level is $p = 0.95$, then $\rho_h[X] \approx VaR_{0.9682}[X]$.

Case 4

We will consider power distortion function $g(x) = x^{\alpha}$, which is a concave distortion function at $0 < \alpha < 1$ and a convex distortion function at $\alpha > 1$, as well as indicator concave distortion function $1_{\{x>1-p\}}$.

Let's consider a distortion function built with this superposition: $1_{\{x>1-p\}}(g(x))$.

However, since $h(x) = 1_{\{x>1-p\}}(g(x)) = 1_{\{g(x)>1-p\}}(x)$ and inequality $x^{\alpha} > 1-p$ is equivalent to inequality $x > (1-p)^{\frac{1}{\alpha}}$, then

$$h(x)=1_{\{x>1-p\}}(g(x))=1_{\{x>(1-p)^{\frac{1}{\alpha}}\}}(x)=1_{\{x>1-(1-(1-p)^{\frac{1}{\alpha}})\}}(x).$$

According to Definition 1, $\rho_h[X]=VaR_{1-(1-p)^{\frac{1}{\alpha}}}[X]$ is distortion risk measure corresponding to the given distortion function, i.e. known risk measure VaR with the confidence level changed in such a way.

The growth of these risk measures with increasing confidence probability strongly depends on the choice of parameter α.

For example, if the initial confidence level is p = 0.95, then at $\alpha=2$, $\rho_h[X]\approx VaR_{0.025}[X]$ this risk measure grows very slowly with increasing confidence probability; at $\alpha=1$, $\rho_h[X]\approx VaR_{0.95}[X]$ it is standard VaR measure; and at $\alpha=\frac{1}{2}$, $\rho_h[X]\approx VaR_{0.9975}[X]$ this risk measure grows rapidly with increasing confidence probability.

Case 5

Let's consider $g(x)=xe^{1-x}$ function, a concave distortion function, as well as indicative concave distortion function $1_{\{x>1-p\}}$.

Let's consider a distortion function built with this superposition: $1_{\{x>1-p\}}(g(x))$.

However, since $h(x)=1_{\{x>1-p\}}(g(x))=1_{\{g(x)>1-p\}}(x)$ and inequality $xe^{1-x}>1-p$ is equivalent to inequality $-xe^{-x}<-\frac{1-p}{e}$, from which it follows that $x>-W(-\frac{1-p}{e})$, where $W(x)$ is the well-known Lambert W-function [22], therefore

$$h(x)=1_{\{x>1-p\}}(g(x))=1_{\{x>-W(-\frac{1-p}{e})\}}(x)=$$

$$=1_{\{x>1-[1+W(-\frac{1-p}{e})]\}}(x).$$

According to Definition 1,
$\rho_h[X]=VaR_{1+W(-\frac{1-p}{e})}[X]$ is distortion risk measure corresponding to the given distortion function, i.e. known risk measure VaR with the confidence level changed in such a way. This risk measure grows fast with increasing confidence probability.

For example, if the initial confidence level is p = 0.95, then $-\frac{1-p}{e}\approx-0.0184$ and then, using the well-known expansion of the Lambert W-function in a power series $|x|<\frac{1}{e}$ converging at

$$W(x)=\sum_{n=1}^{\infty}\frac{(-n)^{n-1}}{n!}x^n=x-x^2+\frac{3}{2}x^3-\frac{8}{3}x^4+\frac{125}{24}x^5-\ldots,$$

we get $W(-0.0184)\approx-0.0187$ and thus, $\rho_h[X]\approx VaR_{0.9813}[X]$.

Case 6

Let's consider $g(x)=\min\{\frac{x}{1-p},1\}$ function, a concave distortion function, as well as indicative concave distortion function $1_{\{x>1-p\}}(x)$.

Let's consider a distortion function built with this superposition: $1_{\{x>1-p\}}(g(x))$.

However,

$$h(x)=1_{\{x>1-p\}}(g(x))=\begin{cases}1, & if \quad x>(1-p)^2\\ 0, & if \quad 0\le x\le(1-p)^2\end{cases}=1_{\{x>(1-p)^2\}}(x).$$

If we introduce concave distortion function $g_2(x)=x^{\frac{1}{2}}$ that belongs to the family of distortion functions studied in Case 4, then $1_{\{x>1-p\}}(g_2(x))=1_{\{x>(1-p)^2\}}(x)$.

Thus, distortion function $h(x)$ can also be represented as the following superposition:

$$h(x)=1_{\{x>1-p\}}(g_2(x))=\begin{cases}1, & if \quad x>(1-p)^2\\ 0, & if \quad 0\le x\le(1-p)^2\end{cases}=$$

$$=1_{\{x>(1-p)^2\}}(x)=1_{\{x>1-(1-(1-p)^2\}}(x).$$

According to Definition 1, $\rho_h[X]=VaR_{1-(1-p)^2}[X]$ is distortion risk measure corresponding to the given distortion function, i.e. known risk measure VaR with the confidence level changed in such a way. This risk measure grows fast with increasing confidence probability.

However, if we recall formula (2) for VaR squared, we get:

$$\rho_h[X] = VaR_p^{(2)}[X].$$

Thus, we found out that new risk measure VaR squared also belongs to the class of distortion risk measures, and it corresponds to the distortion function obtained as a superposition of function $1_{\{x>1-p\}}(x)$ with any distortion function: $g(x) = \min\{\frac{x}{1-p}, 1\}$ or $g_2(x) = x^{\frac{1}{2}}$.

We can prove that the following, more general definition is true.

***Definition* 3**

Risk measure *VaR* to the power of *n* (for any natural *n*) belongs to the class of distortion risk measures and corresponds to the distortion function obtained in any superposition of a function with any distortion function:

$$g(x) = \min\{\frac{x}{1-p}, 1\} \text{ or } g_n(x) = x^{\frac{1}{n}}:$$

$$h(x) = 1_{\{x>1-p\}}(\underbrace{g(g(\ldots(g(x))}_{n-1-\text{times}} = 1_{\{x>1-p\}}(g_n(x)),$$

i.e. $VaR_p^{(n)}[X] = \rho_h[X]$.

Proof

We will consider function $g(x) = \min\{\frac{x}{1-p}, 1\}$, a concave distortion function. The following superposition $\underbrace{g(g(\ldots(g(x))}_{n-1-times}$ also represents a concave distortion function as follows:

$$\underbrace{g(g(\ldots(g(x))}_{n-1-times} = \begin{cases} 1, \; if \;\; x > (1-p)^{n-1} \\ \dfrac{x}{(1-p)^{n-1}}, \; if \;\; 0 \le x \le (1-p)^{n-1} \end{cases},$$

concave distortion function $h(x) = 1_{\{x>1-p\}}(\underbrace{g(g(\ldots(g(x))}_{n-1-times}$ is as follows:

$$h(x) = 1_{\{x>1-p\}}(\underbrace{g(g(\ldots(g(x))}_{n-1-times} =$$

$$\begin{cases} 1, \; if \;\; x > (1-p)^n \\ 0, \; if \;\; 0 \le x \le (1-p)^n \end{cases} = 1_{\{x>(1-p)^n\}}(x).$$

If we introduce concave distortion function $g_n(x) = x^{\frac{1}{n}}$ that belongs to the family of distortion functions studied in Case 4, then $1_{\{x>1-p\}}(g_n(x)) = 1_{\{x>(1-p)^n\}}(x) = h(x)$.

Thus, distortion function *h*(*x*) can also be represented as the following superposition:

$$h(x) = 1_{\{x>1-p\}}(g_n(x)) = 1_{\{x>(1-p)^n\}}(x) = 1_{\{x>1-(1-(1-p)^n\}}(x).$$

According to Definition 1, $\rho_h[X] = VaR_{1-(1-p)^n}[X]$ is distortion risk measure corresponding to the given distortion function, i.e. known risk measure VaR with the confidence level changed in such a way. This risk measure grows fast with increasing confidence probability.

However, if we recall formula (4) for VaR to the power of *n*, we get $\rho_h[X] = VaR_p^{(n)}[X]$.

The definition is proved.

The more general statement is also valid for *VaR* risk measures to any power of $t \ge 1$.

***Definition* 4**

Risk measure *VaR* to the power of *t*, $\mathrm{VaR}_p^{(t)}[X]$ (at any actual $t \ge 1$), where *t* is as follows: $t = k + \alpha$, where *k* is a natural number, and α is a real number, moreover $0 \le \alpha < 1$, it is a distorted risk measure and it is obtained as a risk measure corresponding to the distortion function, which can be represented as superposition of distortion functions $1_{\{x>1-p\}}(x)$,

$$g(x) = \min\{\frac{x}{1-p}, 1\}, \text{ and } g_\alpha(x) = \min\{\frac{x}{1-\alpha p}, 1\},$$

and $g_{k-1}(x) = x^{\frac{1}{k-1}}$ in the following two ways:

$$h(x) = 1_{\{x>1-p\}}(\underbrace{g(g(\ldots(g}_{k-1-times}(g_\alpha(x)) = 1_{\{x>1-p\}}(g_{k-1}(g_\alpha(x))),$$

i.e. $VaR_p^{(t)}[X] = \rho_h[X]$.

Proof

Functions $g(x)=\min\{\frac{x}{1-p},1\}$ and $g_\alpha(x)=\min\{\frac{x}{1-\alpha p},1\}$ are concave distortion functions. Next superposition $\underbrace{g(g(...(g}_{k-1-times}(g_\alpha(x))...)$ also represents a concave distortion function as follows:

$$\underbrace{g(g(...(g}_{k-1-times}(g_\alpha(x))...)=\begin{cases}1, \ if \ \ x>(1-p)^{k-1}(1-\alpha p)\\ \dfrac{x}{(1-p)^{k-1}(1-\alpha p)},\\ if \ \ 0\le x\le(1-p)^{k-1}(1-\alpha p),\end{cases}$$

and concave distortion function $h(x)=1_{\{x>1-p\}}(\underbrace{g(g(...(g}_{k-1-times}(g_\alpha(x)...)$ is as follows:

$$h(x)=$$

$$=\begin{cases}1, \ if \ \ x>(1-p)^k(1-\alpha p)\\ 0, \ if \ \ 0\le x\le(1-p)^k(1-\alpha p)\end{cases}=1_{\{x>(1-p)^k(1-\alpha p)\}}(x).$$

With function $g_{k-1}(x)=x^{\frac{1}{k-1}}$, the distortion function $h(x)$ can also be represented as the following superposition:

$$h(x)=1_{\{x>1-p\}}(g_{k-1}(g_\alpha(x)))=$$
$$=1_{\{x>(1-p)^k(1-\alpha p)\}}(x)=1_{\{x>1-(1-(1-p)^k(1-\alpha p))\}}(x).$$

According to Definition 1, $\rho_h[X]=VaR_{1-(1-p)^k(1-\alpha p)}[X]$ is distortion risk measure corresponding to the given distortion function, i.e. known risk measure VaR with the confidence level changed in such a way.

However, if we recall formula (9) for *VaR* to the power of *t*, we get $\rho_h[X]=VaR_p^{(t)}[X]$.

The definition is proved.

In general, any concave distortion function *g* gives the distribution tail more weight than the identical function *g*(*x*) = *x*, while any convex distortion function *g* gives the tail less weight than the identical function *g*(*x*) = *x* [15]. Therefore, in particular, any concave distortion function *g* gives the distribution tail more weight than any convex distortion function.

It is good to know when building a risk measure with the required properties.

The question is if risk measure $ES_p^{(2)}[X]$ is a distorted risk measure.

Case 6

We will consider function $g(x)=\min\{\frac{x}{1-p},1\}$, a concave distortion function, as well as distortion function built with superposition: $g(g(x))$.

It is easy to check,

$$h(x)=g(g(x))=\begin{cases}1, \ if \ \ x>(1-p)^2\\ \dfrac{x}{(1-p)^2}, \ if \ \ 0\le x\le(1-p)^2\end{cases}$$

and

$$h'(x)=\begin{cases}0, \ if \ \ x>(1-p)^2\\ \dfrac{1}{(1-p)^2}, \ if \ \ 0\le x\le(1-p)^2.\end{cases}$$

According to Theorem 2, the distorted risk measure corresponding to a given distortion function turns out to be a measure that can be represented as follows

$$\rho_h[X]=\int\limits_{[0,(1-p)^2]}VaR_{1-q}[X]\frac{1}{(1-p)^2}dq+$$
$$+\int\limits_{[(1-p)^2,1]}VaR_{1-q}[X]\times 0dq=$$
$$=\frac{1}{(1-p)^2}\int\limits_{[0,(1-p)^2]}VaR_{1-q}[X]dq=$$
$$=\frac{1}{(1-p)^2}\int\limits_{[1-(1-p)^2,1]}VaR_q[X]dq.$$

However, if we recall formula (10) for *ES* squared, we get $\rho_h[X]=ES_p^{(2)}[X]$.

We found out that new risk measure ES squared, introduced in this work, also belongs to the class of distortion risk measures, and it corresponds to the described distortion function.

The question is if risk measure $ES_p^{(n)}[X]$ is a distorted risk measure.

Definition 5

Risk measure ES to the power of *n* (for any natural *n*) belongs to the class of distortion risk measures, and it corresponds to the distortion function obtained as any superposition of functions $g(x)=\min\{\frac{x}{1-p},1\}$ as follows:

$$h(x)=\underbrace{g(g(...(g(x))}_{n-times},\ \text{i.e. } ES_p^{(n)}[X]=\rho_h[X].$$

Proof

Function $g(x)=\min\{\frac{x}{1-p},1\}$ is a concave distortion function. Next superposition $\underbrace{g(g(...(g(x))}_{n-times}$ also represents a concave distortion function as follows:

$$h(x)=\underbrace{g(g(...(g(x))}_{n-times}=\begin{cases}\frac{x}{(1-p)^n},\ if\ \ 0\le x\le(1-p)^n\\ 1,\ if\ \ (1-p)^n<x\le 1\end{cases}$$

and

$$h'(x)=\begin{cases}0,\ if\ \ x>(1-p)^n\\ \frac{1}{(1-p)^n},\ if\ \ 0\le x\le(1-p)^n\end{cases}.$$

According to Theorem 2, the distorted risk measure corresponding to a given distortion function *h*(*x*) turns out to be a measure that can be represented as follows

$$\rho_h[X]=\int_{[0,(1-p)^n]}VaR_{1-q}[X]\frac{1}{(1-p)^n}dq+$$

$$+\int_{[(1-p)^n,1]}VaR_{1-q}[X]\times 0dq=$$

$$=\frac{1}{(1-p)^n}\int_{[0,(1-p)^n]}VaR_{1-q}[X]dq=$$

$$=\frac{1}{(1-p)^n}\int_{[1-(1-p)^n,1]}VaR_q[X]dq.$$

However, if we recall formula (11) for *ES* to the power of *n*, we get $\rho_h[X]=ES_p^{(n)}[X]$.

We found out that new risk measure *ES* to the power of *n* also belongs to the class of distortion risk measures. It corresponds to the described distortion function and is presented as usual risk measure ES with the confidence probability changed in a certain way.

The definition is proved.

The question is if risk measure $ES_p^{(t)}[X]$ is a distorted risk measure.

Definition 6

Risk measure *ES* in power of *t* for any real $t\ge 1$, represented as $t=k+\alpha$, where *k* is a natural number, and *a* is a real number, $0<\alpha<1$, belongs to the class of distortion risk measures, and corresponds to the distortion function obtained as any superposition of functions $g(x)=\min\{\frac{x}{1-p},1\}$ and $g_\alpha(x)=\min\{\frac{x}{1-\alpha p},1\}$ as follows:

$$h(x)=\underbrace{g(g(...(g(}_{k-times}g_\alpha(x))...),\ \text{i.e. } ES_p^{(t)}[X]=\rho_h[X].$$

Proof

Function $g(x)=\min\{\frac{x}{1-p},1\}$ is a concave distortion function. Superposition $\underbrace{g(g(...(g(x))}_{n-times}$ also represents a concave distortion function as follows:

$$h(x)=\underbrace{g(g(...(g(}_{k-times}g_\alpha(x))...)=$$

$$=\begin{cases}\frac{x}{(1-p)^k(1-\alpha p)},\ if\ \ 0\le x\le(1-p)^k(1-\alpha p)\\ 1,\ if\ \ (1-p)^k(1-\alpha p)<x\le 1,\end{cases}$$

and

$$h'(x)=\begin{cases}0,\ if\ \ x>(1-p)^k(1-\alpha p)\\ \frac{1}{(1-p)^k(1-\alpha p)},\ if\ \ 0\le x\le(1-p)^k(1-\alpha p)\end{cases}.$$

According to Theorem 2, the distorted risk measure corresponding to a given distortion function $h(x)$ turns out to be a measure that can be represented as follows:

$$\rho_h[X] = \int\limits_{[0,(1-p)^k(1-\alpha p)]} VaR_{1-q}[X]\frac{1}{(1-p)^k(1-\alpha p)}dq +$$

$$+ \int\limits_{[(1-p)^n,1]} VaR_{1-q}[X]\times 0 dq =$$

$$= \frac{1}{(1-p)^k(1-\alpha p)} \int\limits_{[0,(1-p)^k(1-\alpha p)]} VaR_{1-q}[X]dq =$$

$$= \frac{1}{(1-p)^k(1-\alpha p)} \int\limits_{[1-(1-p)^k(1-\alpha p,1]} VaR_q[X]dq\,.$$

If we recall formula (13) for *ES* to the power of *t*, we get: $\rho_h[X] = ES_p^{(t)}[X]$.

We found out that new risk measure *ES* to the power of *t* also belongs to the class of distortion risk measures. It corresponds to the described distortion function and is presented as usual risk measure ES with the confidence probability changed in a certain way.

The definition is proved.

We will now consider case 7 of two random variables *X* and *Y* with different discrete distribution laws, whose risks do not distinguish between the known risk measures VaR and ES [15]. Generalizing risk measure ES with random values of losses that obey discrete distribution laws has its own specifics. In particular, if the random loss *X* obeys a discrete distribution, then $ES_p[X]$ is expressed through the values of VaR and the expected value of the excess losses over VaR [15]:

$$ES_p[X] = VaR_p[X] + \frac{1-F_X(VaR_p[X])}{1-p}E[X - VaR_p[X] \mid X > VaR_p[X]]. \quad (15)$$

This example by C. Yin and D. Zhu [15] shows that risk measures $VaR_p[X]$ and $ES_p[X]$ may not distinguish between the risks created by *X* and *Y*. At the same time, an example of a certain risk measure that distinguishes between their risks is given. This measure coincides with risk measure $ES_p^{(2)}[X]$ introduced in this work.

Case 7

Let us consider two random variables *X* and *Y* that simulate risks with distribution functions, respectively:

$$F_X(x) = \begin{cases} 0, if \;\; x < 0, \\ 0.6, if \;\; 0 \le x < 100 \\ 0.975, if \;\; 100 \le x < 500 \\ 1, \; if \;\; x \ge 500 \end{cases}$$

and

$$F_Y(x) = \begin{cases} 0, if \;\; x < 0, \\ 0.6, if \;\; 0 \le x < 100 \\ 0.99, if \;\; 100 \le x < 1100 \\ 1, \; if \;\; x \ge 1100 \end{cases}$$

It is easy to check that $E(X) = E(Y) = 50$, $VaR_{0.95}[X] = VaR_{0.96}[X] = 100$, $VaR_{0.95}[Y] = VaR_{0.96}[Y] = 100$.

ES can be calculated by formula (15) and we get:

$$ES_{0.95}[X] = ES_{0.95}[Y] = 300,$$

$ES_{0.96}[X] = ES_{0.96}[Y] = 350$. When $p = 0.95$ and $p = 0.96$, then according to the risk measures VaR and ES, both *X* and *Y* have the same risk! However, the maximum loss for *Y* (1100) more than doubles the loss for *X* (500), and it is clear that risk *Y* is greater than risk *X*.

We now consider distortion measure ρ_h with distortion function искажения $h(x) = g(g(x))$ and

$$g(x) = \begin{cases} \dfrac{x}{1-p}, if \;\; 0 \le x \le 1\text{-}p \\ 1, if \;\; 1\text{-}p < x \le 1, \end{cases}$$

The, according to case 6,

$$\rho_h[X] = \frac{1}{(1-p)^2} \int\limits_{[1-(1-p)^2,1]} VaR_q[X]dq = ES_p^{(2)}[X].$$

And numerically for $p = 0.95$

$$\rho_h[X] = \frac{1}{(0.05)^2} \int_{[1-0.05^2,1]} VaR_q[X]dq = ES^{(2)}_{0.95}[X],$$

i.e.

$$\rho_h[X] =$$

$$= \frac{1}{0.0025} \int_{[0.9975,1]} VaR_q[X]dq = \frac{500}{0.0025}(1-0.9975) = 500$$

and

$$\rho_h[Y] =$$

$$= \frac{1}{0.0025} \int_{[0.9975,1]} VaR_q[X]dq = \frac{1100}{0.0025}(1-0.9975) = 1100.$$

Then at $p = 0.95$, $\rho_h[X] = ES^{(2)}_{0.05}[X] = 500$ and $\rho_h[Y] = ES^{(2)}_p[Y] = 1100.$

In this case, risk measure $\rho_h = ES^{(2)}_p$, distinguishing between different risk levels for *X* and *Y*, turned out to be more suitable for risk management than usual risk measures VaR and ES.

CONCLUSIONS

A vigorous theoretical study of a class of distortion risk measures took place in the last decade. They have recently become widespread in financial and insurance applications due to their attractive properties. In his earlier works, the author introduced and investigated risk measures "*VaR* to the power of *t*" that allow identifying various financial catastrophic risks. In this paper, the author described and developed a composite method for creating a new class of distortion functions and corresponding distortion risk measures. By this method, the author proves that risk measures "*VaR* to the power of *t*" belong to the class of distortion risk measures, and describes the corresponding distortion functions. Also, the author introduces a new class of risk measures "ES to the power of *t*", proves that they also belong to the class of distortion risk measures and describes the corresponding distortion functions. Various cases illustrate the relevant concepts and results that demonstrate the importance of "*VaR* to the power of *t*" and "ES to the power of *t*" risk measures as subsets of distortion risk measures identifying various financial catastrophic risks. Distortion risk measures are currently well studied and have many useful and convenient properties. Thus, all the properties possessed by the distortion risk measures [12] are also possessed by the families of measures "*VaR* to the power of *t*" and "ES to the power of *t*".

ACKNOWLEDGEMENTS

This article is based on the budgetary-supported research 4.10 "Researching Methods for Measuring Risks at the Corporate and Macrofinancial Levels" according to the state task carried out by the Higher School of Finance and Management, Russian Presidential Academy of National Economy and Public Administration, Moscow, Russia.

REFERENCES

1. Crouhy M., Galai D., Mark R. The essentials of risk management. New York: McGraw-Hill Book Co.; 2006. 414 p. (Russ. ed.: Crouhy M., Galai D., Mark R. Osnovy risk-menedzhmenta. Moscow: Urait; 2006. 414 p.).
2. Hull J.C. Risk management and financial institutions. New York: Pearson Education International; 2007. 576 p.
3. Jorion P. Value at risk: The new benchmark for managing financial risk. New York: McGraw-Hill Education; 2007. 624 p.
4. Wang S.S. A class of distortion operators for pricing financial and insurance risks. *The Journal of Risk and Insurance*. 2000;67(1):15–36. DOI: 10.2307/253675
5. Szegö G., ed. Risk measures for the 21st century. Chichester: John Wiley & Sons Ltd; 2004. 491 p.
6. Minasyan V.B. A new risk measure VaR squared and its calculation. The case of uniform and triangular loss distributions. *Upravlenie finansovymi riskami = Financial Risk Management Journal.* 2019;(3):200–208. (In Russ.).

7. Minasyan V.B. A new risk measure VaR squered and its calculation. The case of the general law of loss distribution, comparison with other risk measures. *Upravlenie finansovymi riskami = Financial Risk Management Journal.* 2019;(4):298–320. (In Russ.).
8. Minasyan V.B. New ways to measure catastrophic financial risk: "VaR to the power of t" measures and how to calculate them. *Finansy: teoriya i praktika = Finance: Theory and Practice.* 2020;24(3):92–109. (In Russ.). DOI: 10.26794/2587–5671–2020–24–3–92–109
9. Denneberg D. Non-additive measure and integral. Dordrecht: Kluwer Academic Publishers; 1994. 178 p. (Theory and Decision Library B. Vol. 27). DOI: 10.1007/978–94–017–2434–0
10. Wang S., Dhaene J. Comonotonicity, correlation order and premium principles. *Insurance: Mathematics and Economics*. 1998;22(3):235–242. DOI: 10.1016/S 0167–6687(97)00040–1
11. Artzner P., Delbaen F., Eber J.-M., Heath D. Coherent measures of risk. *Mathematical Finance*. 1999;9(3):203–228. DOI: 10.1111/1467–9965.00068
12. Denuit M., Dhaene J., Goovaerts M., Kaas R. Actuarial theory for dependent risks: Measures, orders and models. Chichester: John Wiley & Sons Ltd; 2005. 440 p. DOI: 10.1002/0470016450
13. Zhu L., Li H. Tail distortion risk and its asymptotic analysis. *Insurance: Mathematics and Economics*. 2012;51(1):115–121. DOI: 10.1016/j.insmatheco.2012.03.010
14. Yang F. First- and second-order asymptotics for the tail distortion risk measure of extreme risks. *Communications in Statistics — Theory and Methods*. 2015;44(3):520–532. DOI: 10.1080/03610926.2012.751116
15. Yin C., Zhu D. New class of distortion risk measures and their tail asymptotics with emphasis on Va R. *Journal of Financial Risk Management*. 2018;7(1):12–38. DOI: 10.4236/jfrm.2018.71002
16. Belles-Sampera J., Guillén M., Santolino M. Beyond value-at-risk: GlueVaR distortion risk measures. *Risk Analysis*. 2014;34(1):121–134. DOI: 10.1111/risa.12080
17. Belles-Sampera J., Guillén M., Santolino M. GlueVaR risk measures in capital allocation applications. *Insurance: Mathematics and Economics*. 2014;58:132–137. DOI: 10.1016/j.insmatheco.2014.06.014
18. Cherubini U., Mulinacci S. Contagion-based distortion risk measures. *Applied Mathematics Letters*. 2014;27:85–89. DOI: 10.1016/j.aml.2013.07.007
19. Dhaene J., Kukush A., Linders D., Tang Q. Remarks on quantiles and distortion risk measures. *European Actuarial Journal*. 2012;2(2):319–328. DOI: 10.1007/s13385–012–0058–0
20. Wang S.S. Premium calculation by transforming the layer premium density. *ASTIN Bulletin: The Journal of the IAA*. 1996;26(1):71–92. DOI: 10.2143/AST.26.1.563234
21. Artzner P., Delbaen F., Eber J.-M., Heath D. Thinking coherently. *Risk*. 1997;10(11):68–71.
22. Corless R.M., Gonnet G.H., Hare D.E.G., Jeffrey D.J., Knuth D.E. On the Lambert W function. *Advanced Computational Mathematics*. 1996;5(1):329–359. DOI: 10.1007/BF02124750

ABOUT THE AUTHOR

Vigen B. Minasyan — Cand. Sci. (Phis.-Math.), Assoc. Prof., Head of Limitovskii corporate finance, investment design and evaluation department, Higher School of Finance and Management, Russian Presidential Academy of National Economy and Public Administration, Moscow, Russia
minasyanvb@ranepa.ru, minasyanvb@yandex.ru

The article was submitted on 02.10.2020; revised on 16.10.2020 and accepted for publication on 22.10.2020.
The author read and approved the final version of the manuscript.