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New Risk Measures for Variance Distortion and Catastrophic Financial Risk Measures

V.B. Minasyan
Russian Presidential Academy of National Economy and Public Administration, Moscow, Russia
https://orcid.org/0000-0001-6393-145X

ABSTRACT

In recent years, expectation distortion risk measures have been widely used in financial and insurance applications due to their attractive properties. The author introduced two new classes of financial risk measures "*VaR* raised to the power of *t*" and "*ES* raised to the power of *t*" in his works and also investigated the issue of the belonging of these risk measures to the class of risk measures of expectation distortion, and described the corresponding distortion functions. The **aim** of this study is to introduce a new concept of variance distortion risk measures, which opens up a significant area for investigating the properties of these risk measures that may be useful in applications. The paper proposes a **method** of finding new variance distortion risk measures that can be used to acquire risk measures with special properties. As a **result** of the study, it was found that the class of risk measures of variance distortion includes risk measures that are in a certain way related to "*VaR* raised to the power of *t*" and "*ES* raised to the power of *t*" measures. The article describes the composite method for constructing new variance distortion functions and corresponding distortion risk measures. This **method** is used to build a large set of examples of variance distortion risk measures that can be used in assessing certain financial risks of a catastrophic nature. The author **concludes** that the study of the variance distortion risk measures introduced in this paper can be used both for the development of theoretical risk management methods and in the practice of business risk management in assessing unlikely risks of high catastrophe.

Keywords: catastrophic financial risks; expectation distortion risk measures; variance distortion risk measures, distortion functions; composite method; coherent financial risk measures; "*VaR* raised to the power of *t*" risk measures; "*ES* raised to the power of *t*" risk measures

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INTRODUCTION

The risk measure shall be designated as the mapping ρ of the set of random variables *X*, associated with the risk portfolios of assets and / or liabilities (the resultant variables of these portfolios) into the real line *R*. In the following discussion, *X* will be represented as the value of the corresponding losses, i.e. positive values of the *X* variables will represent losses, while negative values represent gains.

Expectation distortion risk measures represent a special and important group of risk measures that are widely used in finance and insurance as the calculation of capital requirements and the principles for calculating indicators related to "risk appetite" for the regulator and company executive. Several popular risk measures have proven to belong to the family of expectations distortion risk measures. For example, value at risk (*VaR*), tail value at risk or expected shortfall (*ES*) (see, for example, [1–3]), and S.S. Wang distortion risk measure [4]. Expectation distortion risk measures satisfy the most important properties that a "good" risk measure should have, including positive homogeneity, translational invariance, and monotonicity (see, for example, [5]).

As proved by D. Denneberg and S. Wang c J. Dhaene [6, 7], when the corresponding distortion function is concave, the distortion risk measure is also subadditive. *VaR* is one of the most popular risk measures used in risk management and banking supervision due to its computational simplicity and for some regulatory reasons, despite its shortcomings as a risk measure. For example, *VaR* is not a subadditive risk measure (see, for example, [8, 9]). The *ES* risk measure, being coherent (see, for example, [2, 3]), is interested only in losses exceeding *VaR* and ignores useful information about the distribution of losses below *Va* R.

L. Zhu and H. Li [10] presented and studied the distortion risk measure, which was reformulated by F. Yang [11].

C. Yin, D. Zhu [12] in particular, described three methods for constructing distortion risk measures: composite, mixing method and an approach based on copula (connective) theory.

Many researchers have proposed new classes of distortion measures. For example, as an extension of *VaR* and *ES*, J. Belles-Sampera, M. Guillén, M. Santolino [13] proposed a new class of distortion risk measures called risk measures Glue*VaR*, which can be expressed as a combination of *VaR* and *ES* indicators at different levels of confidence. They obtained closed-form analytical expressions for these measures with the most commonly used distribution functions in finance and insurance. The application of Glue*VaR* risk measures related to capital allocation was discussed in article [14].

V.B. Minasyan [15] introduced the *VaR* to the power of *t* risk measures, and in [16] it was proved that the family of measures *VaR* to the power of *t* is a subset of the set of risk measures for the expectation distortion. Thus, any measure of risk *VaR* to the power of *t*, for any $t \geq 1$, for any, is an expectation distortion risk measure with a certain distortion function. At the same time, this distortion function was presented.

In the latter work, a family of new risk measures was also introduced, called risk measures "*ES* to the power of *t*" ($ES_p^{(t)}[X]$), or any confidence probability *p* and any real $t \geq 1$. The work investigated the relationship between two classes of risk measures: expectation distortion risk measures and *ES* to the power of *t* risk measures it was proved that the family of measures *ES* to the power of *t* is a subset of the set of expectation distortion risk measures. That is, any *ES* to the power of *t* risk measure, for any $t \geq 1$, is a measure of the risk of expectation distortion with a certain distortion function. Moreover, this distortion function was presented.

Obviously, it is difficult to believe that there is a unique risk measure that can encompass all characteristics of risk. There is no such ideal measure. Moreover, since virtually every risk measure has one number associated with it, each risk measure cannot exhaust all the information about the risk. The families of risk measures *VaR* to the power of *t* and *ES* to the power of *t*, as shown in the works of V.B. Minasyan [15, 16], make it possible to study the right tail of the distribution of losses with any accuracy required for a given case, i.e. examine the tail of the distribution as thoroughly as necessary under the circumstances. In general, during the research process, it is advisable to look for risk measures that are ideal for a particular problem. Since all the proposed risk measures are erroneous and limited in their application, the choice of the appropriate risk measure continues to be a hot topic in risk management.

In light of this, the development of new directions for the detection of new risk measures that have the ability to more accurately assess specific types of catastrophic risks, considering all kinds of necessary properties of such measures, seems legitimate. In this paper, an attempt is made to propose a new direction in the search for such measures with an appropriate methodology for their search. We propose a new concept for measuring the risk of variance distortion, which opens up a new area of such a search.

Distortion functions

The distortion function is a non-decreasing function g: [0,1] → [0,1] such that, g (0) = 0, g (1) = 1. Many distortion functions *g* have already been proposed in the literature. A summary of the various distortion functions used to construct expectation distortion risk measures can be found in [9, 16].

Expectation distortion risk measures

Let (Ω, *F, P*) — be a probability space on which all random variables representing the risks of interest to us are defined. Let F_x — be the integral distribution function of a random variable *X*, and the dual distribution function we denote as $\overline{F}_X$, i.e. $\overline{F}_X = 1 - F(x) = P\{X > x\}$. Let *g* be a distortion function.

Expectation distortion of a random variable X is denoted $\rho_g^E[X]$ and defined as

$$\rho_g^E[X] = \int_0^{+\infty} g(\overline{F}_X(x))dx + \int_{-\infty}^{0} [g(\overline{F}_X(x)) - 1]dx, \quad (1)$$

provided that at least one of the two integrals indicated above is finite. If *X* is a non-negative random variable, then ρ_g^E simplifies to

$$\rho_g^E[X] = \int_0^{+\infty} g(\overline{F}_X(x))dx.$$

It should be noted that this definition implies that in the case when the distortion function is an identical function, i.e. $g(x)=x$, then, and it is easy to check, the skewed expectation is the same as the normal expectation: $\rho_g^E[X]=E[X]$.

Due to the fact that the expected value of a random variable is considered the most important way of assessing the future value of a random variable *X*, it is natural to assume that, since risks arise due to one or another deviation of the value of a random variable from its expected value, the corresponding risk measures can be modeled as corresponding "distortion" of the expected value with the appropriate distortion function.

The distorted expectation $\rho_g^E[X]$ is called *the expectation distortion risk measure with the distortion function g* (see, for example, [17]).

As noted in [9], the well-known risk measure *VaR* (see, for example, [1–3]) is an expectation distortion risk measure corresponding to the distortion function $g(x)=1_{\{x>1-p\}}$, $p\in(0,1)$, $\rho_g^E[X]=\mathrm{VaR}_p[X]$.

Expectation distortion risk measures are a special class of risk measures that were introduced by D. Denneberg [6] and revised by S.S. Wang [4, 18].

Expectation distortion risk measures satisfy a variety of properties, including positive homogeneity, translation invariance, and monotonicity.

It is known (see [17]) that another measure of risk after *VaR*, which is represented as an expectation distortion risk measure, is the well-known *ES* measure — a measure of the expected deficit, conditional *VaR* (see, for example, [1–3]). The corresponding distortion function is $g(x)=\min\{\frac{x}{1-p},1\}$, $p\in[0,1]$, and under the assumption of the continuity of the distribution function F_x the corresponding expectation distortion risk measure is

$$\rho_g^E[X]=\mathrm{ES}_p[X].$$

V.B. Minasyan [16] proved (see *Statement 4*) that the risk measures *VaR* to the power of *t*, introduced by him in [15], $\mathrm{VaR}_p^{(t)}[X]$ for any real number $t\geq 1$ are risk expectation distortion risk measures, and the corresponding distortion function can be described as follows.

We represent the number *t* as: $t=k+\alpha$, where *k* — natural number α — a real number, with $0\leq\alpha<1$. Then risk measure $VaR_p^{(t)}[X]$ will be an expectation distortion risk measure, which can be represented as a superposition of distortion functions

$$1_{\{x>1-p\}}(x),\ g(x)=\min\{\frac{x}{1-p},1\}$$

$$\text{и } g_\alpha(x)=\min\{\frac{x}{1-\alpha p},1\} \text{ и } g_{k-1}(x)=x^{\frac{1}{k-1}}$$

in two ways:

$$h(x)=1_{\{x>1-p\}}(\underbrace{g(g(\ldots(g}_{k-1-\text{times}}(g_\alpha(x))=1_{\{x>1-p\}}(g_{k-1}(g_\alpha(x))),$$

i.e.

$$VaR_p^{(t)}[X]=\rho_h^E[X].$$

It was also proved in [16] (see *Statement 4*) that the introduced risk measures *ES* to the power of *t*, $ES_p^{(t)}[X]$ for any real $t\geq 1$ are expectation distortion risk measures, and the corresponding distortion function can be described as follows.

We represent the number *t* as: $t=k+\alpha$, where *k* — natural number and α — is a real number, with $0\leq\alpha<1$, then the risk measure $ES_p^{(t)}[X]$ will be an expectation distortion risk measure, and it is obtained as a risk measure corresponding to the distortion function obtained as a superposition of functions

$$g(x)=\min\{\frac{x}{1-p},1\}$$

and a function $g_\alpha(x)=\min\{\frac{x}{1-\alpha p},1\}$ of the following form:

$$h(x)=\underbrace{g(g(\ldots(g(}_{k-\text{раз}}g_\alpha(x)),$$

$$\text{i.e. } ES_p^{(t)}[X]=\rho_h^E[X].$$

Variance distortion risk measures

The most established measure of the risk of any risk factor, which is a certain random variable *X*, is the variance of this value (or its standard deviation). Expectations distortion risk measures have arisen by "distorting" the expected value of *X*, and the study of this class of measures has led to significant progress in methods for assessing catastrophic risk measures. The question arises: is it possible to propose to "distort" the variance in a certain way with the hope that this approach will generate a new class of measures, which could be called variance distortion risk measures. We hope that they will have a rich structure that allows one to find risk measures in it that meet certain needs of risk managers and are not satisfied with other classes of risk measures.

It should be noted that this definition should be such that in the case when the distortion function is an identical function, i.e. $g(x)=x$, the distorted value of the variance, which we will denote as ρ_g^D, coincides with the usual variance of a random variable, i.e. $\rho_g^D[X]=D[X]$.

To bring the variance to a form convenient for its "distortion", we transform its well-known expression:

$$D[X]=\int_{-\infty}^{+\infty}(x-E[X])^2 dF_X(x).$$

The transformation below is valid under the following assumptions:

A) $\lim\limits_{x\to+\infty} x^2(1-F_X(x))=0$.

B) $\lim\limits_{x\to-\infty} x^2 F_X(x)=0$.

Assumption A) means that $F_X(x)\to 1$ at $x\to+\infty$ with a sufficiently high speed. For continuous distributions, always $F_X(x)\to 1$ at $x\to+\infty$. But here it is necessary that $F_X(x)$ approaching 1 occurs faster than x^2 approaching infinity.

Assumption B) means that $F_X(x)\to 0$ at $x\to-\infty$ with a sufficiently high speed. For continuous distributions, always $F_X(x)\to 0$ at $x\to-\infty$. But here it is necessary that $F_X(x)$ approaching 0 occurs faster than x^2 approaching infinity.

Using integration by parts and assumptions A) and B), we have:

$$D[X]=-\int_{E[X]}^{+\infty}(x-E[X])^2 d(1-F_X(x))+$$

$$+\int_{-\infty}^{E[X]}(x-E[X])^2 dF_X(x)=$$

$$-(x-E[X])^2(1-F_X(x))\big|_{E[X]}^{+\infty}+$$

$$+2\int_{E[X]}^{+\infty}(1-F_X(x))(x-E[X])dx+$$

$$+(x-E[X])^2 F_X(x)\big|_{-\infty}^{E[X]}-$$

$$-2\int_{-\infty}^{E[X]}F_X(x)(x-E[X])dx=$$

$$=2\int_{E[X]}^{+\infty}\bar{F}_X(x))(x-E[X])dx+$$

$$+2\int_{-\infty}^{E[X]}[\bar{F}_X(x)-1](x-E[X])dx.$$

Based on the last expression, it is quite natural to introduce the following definition. Let g be a distortion function.

The distorted variance of the random variable X, corresponding to the distortion function g, is denoted as $\rho_g^D[X]$ and defined as

$$\rho_g^D[X]=2\int_{E[X]}^{+\infty}g(\bar{F}_X(x))(x-E[X])dx+$$

$$+2\int_{-\infty}^{E[X]}[g(\bar{F}_X(x))-1](x-E[X])dx. \quad (2)$$

provided that at least one of the two integrals above is finite.

It should be noted that when the distortion function is identical, i.e., $g(x)=x$, then the distorted variance coincides with the usual variance: $\rho_g^D[X]=D[X]$.

We will call the distorted variance $\rho_g^D[X]$ as *the variance distortion risk measure with the distortion function g.*

Using definition (2), it is easy to check that the variance distortion risk measure with any distortion function g from a constant (not random) value X = const = is equal to zero. That is, $\rho_g^D[c]=0$.

Search for risk measures from the class of risk measures for variance distortion

We will now look for measures of risk that are contained in various measures of risk of variance distortion.

We will seek appropriate measures by choosing a certain distortion function and obtaining a computational formula for the risk measure of variance distortion corresponding to a given distortion function.

Concave distortion function

$$g(x)=1_{\{x>1-p\}},\quad p\in(0,1).$$

This distortion function in the set of expectation distortion risk measures led to the measurement of risk *VaR* (see [13]). What degree of risk will this lead to when constructing an appropriate measure of the risk of variance distortion?

Hereinafter, we will assume the continuity of the distribution function of the random variable *X*, which represents the corresponding risk factor.

According to formula (2), we have:

$$\rho_g^D[X]=2\int_{E[X]}^{+\infty} 1_{\{\bar{F}_X(x)>1-p\}}(x-E[X])dx+$$

$$+2\int_{-\infty}^{E[X]} [1_{\{\bar{F}_X(x)>1-p\}}-1](x-E[X])dx=$$

$$=2\int_{E[X]}^{+\infty} 1_{\{F_X(x)\le p\}}(x-E[X])dx+$$

$$2\int_{-\infty}^{E[X]} [1_{\{F_X(x)\le p\}}-1](x-E[X])dx.$$

Denoting by the F_X^{-1} function, inverse to the distribution function F_X, we obtain:

$$\rho_g^D[X]=2\int_{E[X]}^{+\infty} 1_{\{x\le F_X^{-1}(p)\}}(x-E[X])dx+$$

$$+2\int_{-\infty}^{E[X]} [1_{\{x\le F_X^{-1}(p)\}}-1](x-E[X])dx. \quad (3)$$

In the further derivation of the formula for $\rho_g^D[X]$ we will have to consider two cases.

A) We assume that $F_X^{-1}(p)<E[X]$, i.e. $VaR_p[X]<E[X]$.

In this case, it is obvious that the first integral in formula (3) is equal to zero. And we obtain:

$$\rho_g^D[X]=2\int_{-\infty}^{E[X]} [1_{\{x\le F_X^{-1}(p)\}}-1](x-E[X])dx=$$

$$=-2\int_{F_X^{-1}(p)}^{E[X]}(x-E[X])dx=-(x-E[X])^2\Big|_{F_X^{-1}(p)=VaR_p[X]}^{E[X]}=$$

$$=(VaR_p[X]-E[X])^2.$$

We consider the second case.

B) We assume that $F_X^{-1}(p)\ge E[X]$, i.e. $VaR_p[X]\ge E[X]$.

In this case, it is obvious that the second integral in formula (3) is equal to zero. And we obtain:

$$\rho_g^D[X]=2\int_{E[X]}^{+\infty} 1_{\{x\le F_X^{-1}(p)\}}(x-E[X])dx=$$

$$=2\int_{E[X]}^{F_X^{-1}(p)}(x-E[X])dx=(x-E[X])^2\Big|_{E[X]}^{F_X^{-1}(p)=VaR_p[X]}=$$

$$=(VaR_p[X]-E[X])^2.$$

Thus, we have proved the following statement.

Statement 1

A variance distortion risk measure, corresponding to the distortion function $g(x)=1_{\{x>1-p\}}$, $p\in(0,1)$, is the risk measure

$$\rho_g^D[X]=(VaR_p[X]-E[X])^2.$$

Note that the value $\tilde{\rho}_g^D[X]=\sqrt{\rho_g^D[X]}$ can also serve as a measure of risk, and its dimension, in contrast to $\rho_g^D[X]$, coincides with the dimension of the random variable *X*.

Obviously,

$$\tilde{\rho}_g^D[X]=|VaR_p[X]-E[X]|=|VaR_p^{rel}[X]|,$$

where through $VaR_p^{rel}[X]$ here denotes the relative value of *VaR*, i.e. the value of the maximum possible unfavorable deviation of a random variable *X* with a given probability *p*.

Concave distortion function

$$g(x)=\min\{\frac{x}{1-p},1\},\ p\in[0,1].$$

This distortion function in the set of risk measures for the distortion of expectations led to the *ES* risk measure (see [17]). Interestingly, to what degree of risk will it lead, applied to construct the corresponding risk measure of variance distortion?

To use formula (2), we first transform the expression $g(\bar{F}_X(x))$. We have:

$$g(\bar{F}_X(x))=$$

$$=\min\{\frac{\bar{F}_X(x)}{1-p},1\}=\begin{cases}\frac{\bar{F}_X(x)}{1-p}, if\ \bar{F}_X(x)\le 1-p,\\ 1, if\ \bar{F}_X(x)>1-p\end{cases}$$

or

$$g(\bar{F}_X(x))=\begin{cases}\frac{1-F_X(x)}{1-p}, if\ F_X(x)>p\\ 1, if\ F_X(x)\le p,\end{cases}$$

which means,

$$g(\overline{F}_X(x)) = \begin{cases} \dfrac{1-F_X(x)}{1-p}, if \; x > F_X^{-1}(p) \\ 1, if \; x \le F_X^{-1}(p). \end{cases}$$

In the further derivation of the formula for $\rho_g^D[X]$ we will have to consider two cases.

A) We assume that $F_X^{-1}(p) < E[X]$, i. e. $VaR_p[X] < E[X]$.

In this case, the first integral in formula (2) has the form:

$$2\int_{E[X]}^{+\infty} g(\overline{F}_X(x))(x-E[X])dx =$$

$$= 2\int_{E[X]}^{+\infty} \frac{1-F_X(x)}{1-p}(x-E[X])dx.$$

The second integral in formula (2) can be transformed as follows:

$$2\int_{-\infty}^{E[X]} [g(\overline{F}_X(x))-1](x-E[X])dx =$$

$$= 2\int_{F_X^{-1}(p)}^{E[X]} [\frac{1-F_X(x)}{1-p}-1](x-E[X])dx.$$

Thus, according to formula (2), we obtain:

$$\rho_g^D[X] = 2\int_{E[X]}^{+\infty} \frac{1-F_X(x)}{1-p}(x-E[X])dx +$$

$$+2\int_{F_X^{-1}(p)}^{E[X]} [\frac{1-F_X(x)}{1-p}-1](x-E[X])dx =$$

$$= 2\int_{F_X^{-1}(p)}^{+\infty} \frac{1-F_X(x)}{1-p}(x-E[X])dx -$$

$$2\int_{F_X^{-1}(p)}^{E[X]} (x-E[X])dx =$$

$$= \frac{1}{1-p}\int_{F_X^{-1}(p)}^{+\infty} (1-F_X(x))d(x-E[X])^2 - (x-E[X])^2 \Big|_{F_X^{-1}(p)}^{E[X]}.$$

Then, using integration by parts, we obtain:

$$\rho_g^D[X] = \frac{1}{1-p}(x-E[X])^2(1-F_X(x))\Big|_{F_X^{-1}}^{+\infty} +$$

$$+\frac{1}{1-p}\int_{F_X^{-1}(p)}^{+\infty} (x-E[X])^2 dF_X(x) + (F_X^{-1}(p)-E[X])^2.$$

Using condition A), we obtain:

$$\rho_g^D[X] = -\frac{1}{1-p}(F_X^{-1}(p)-E[X])^2(1-F_X(F_X^{-1}(p))) +$$

$$+(F_X^{-1}(p)-E[X])^2 +$$

$$+\frac{1}{1-p}\int_{F_X^{-1}(p)}^{+\infty} (x-E[X])^2 dF_X(x). \qquad (4)$$

Using the obvious relation $F_X(F_X^{-1}(p)) = p$, it is easy to see that the sum of the first two terms in formula (4) is equal to zero, which means that the formula is correct:

$$\rho_g^D[X] = \frac{1}{1-p}\int_{F_X^{-1}(p)}^{+\infty} (x-E[X])^2 dF_X(x).$$

Now we consider the second case.

B) We assume that $F_X^{-1}(p) > E[X]$, i. e. $VaR_p[X] > E[X]$.

In this case, obviously, the second integral in formula (2) is equal to zero, i.e.

$$2\int_{-\infty}^{E[X]} [g(\overline{F}_X(x))-1](x-E[X])dx = 0.$$

Therefore, according to formula (2), we have:

$$\rho_g^D[X] = 2\int_{E[X]}^{+\infty} g(\overline{F}_X(x))(x-E[X])dx =$$

$$= 2\int_{E[X]}^{F_X^{-1}(p)} (x-E[X])dx +$$

$$+2\int_{F_X^{-1}(p)}^{+\infty} \frac{1-F_X(x)}{1-p}(x-E[X])dx =$$

$$= (x-E[X])^2 \Big|_{E[X]}^{F_X^{-1}(p)} +$$

$$+\frac{1}{1-p}\int_{F_X^{-1}(p)}^{+\infty} (1-F_X(x))d(x-E[X])^2.$$

Then, using integration by parts, we obtain:

$$\rho_g^D[X]=$$

$$=(F_X^{-1}(p)-E[X])^2+\frac{1}{1-p}(x-E[X])^2(1-F_X(x))\Big|_{F_X^{-1}(p)}^{+\infty}+$$

$$+\frac{1}{1-p}\int_{F_X^{-1}(p)}^{+\infty}(x-E[X])^2 dF_X(x).$$

Using condition A), we obtain:

$$\rho_g^D[X]=$$

$$=(F_X^{-1}(p)-E[X])^2-\frac{1}{1-p}(F_X^{-1}(p)-E[X])^2(1-F_X(F_X^{-1}(p)))+$$

$$+\frac{1}{1-p}\int_{F_X^{-1}(p)}^{+\infty}(x-E[X])^2 dF_X(x).$$

Using the obvious relation $F_X(F_X^{-1}(p))=p$, it is easy to see that the sum of the first two terms in formula (4) is equal to zero, which means that the formula is valid:

$$\rho_g^D[X]=\frac{1}{1-p}\int_{F_X^{-1}(p)}^{+\infty}(x-E[X])^2 dF_X(x). \qquad (5)$$

Thus, we have proved that in all cases this measure of the risk of variance distortion is represented by formula (5).

This formula can be written in the following form:

$$\rho_g^D[X]=\frac{1}{1-p}\int_{VaR_p[X]}^{+\infty}(x-E[X])^2 dF_X(x). \qquad (6)$$

Remembering the variance formula:

$$D[X]=E[(X-E[X])^2]=\int_{-\infty}^{+\infty}(x-E[X])^2 dF_X(x)$$

and comparing it with formula (6), and also, considering that, $P\{X>VaR_p[X]\}=1-p$, we obtain the following representation for this risk measure:

$$\rho_g^D[X]=E[(X-E[X])^2\mid X>VaR_p[X]], \qquad (7)$$

where E[Y|A] denotes the conditional expected value of the random variable Y, subject to the implementation of the random event A.

Or, if you define the conditional variance, provided $X>VaR_p[X]$ by the expression:

$$D[X\mid X>VaR_p[X]]=E[(X-E[X])^2\mid X>VaR_p[X]],$$

we obtain the following representation for this variance distortion risk measure:

$$\rho_g^D[X]=D[X\mid X>VaR_p[X]]. \qquad (8)$$

Thus, we have proved the following statement.

Statement 2

the variance distortion risk measure corresponding to the distortion function

$g(x)=\min\{\frac{x}{1-p},1\},\ p\in[0,1]$, is the risk measure

$$\rho_g^D[X]=\frac{1}{1-p}\int_{VaR_p[X]}^{+\infty}(x-E[X])^2 dF_X(x);$$

b) this risk measure can also be represented as

$$\rho_g^D[X]=D[X\mid X>VaR_p[X]],$$

where

$$D[X\mid X>VaR_p[X]]=E[(X-E[X])^2\mid X>VaR_p[X]].$$

That is, this measure of the risk of losses represents the conditional variance of the random factor X, which represents a risk, provided that the value of these losses exceeded the value $VaR_p[X]$.

As known, the *ES* risk measure, in the case of the continuity of the distribution function of the random variable *X*, can also be represented in two ways:

$$ES_p[X]=\frac{1}{1-p}\int_p^1 VaR_q[X]dq \qquad (9)$$

and

$$ES_p[X]=E[X\mid X>VaR_p[X]]. \qquad (10)$$

Comparing formula (10) and the presentation of our new risk measure for variance distortion $\rho_g^D[X]$ in section b) *Statement 2*, we see that in the class of risk measures for variance distortion, the new risk measure $\rho_g^D[X]$ the same significance as the measure $ES_p[X]$ in the class of expectation distortion risk measures.

Hence, we can conclude that the significance of this measure for the theory and practice of risk management can be no less than the significance of the *ES* risk measure.

And yet, looking at formula (9), we would like to have a formula for our variance distortion risk measure $\rho_g^D[X]$ in a form similar to formula (9) for the *ES* risk measure.

The following proposition can be proved.

Statement 3

The variance distortion risk measure $\rho_g^D[X]$, corresponding to the distortion function

$g(x) = \min\{\frac{x}{1-p}, 1\},\ p \in [0,1]$, can be represented as:

$$\rho_g[X] = \frac{1}{1-p}\int_p^1 (VaR_q^{rel}[X])^2 dq, \quad (11)$$

where

$VaR_q^{rel}[X] = VaR_q[X] - E[X]$, the value of the corresponding relative risk measure *Va* R.

Proof

According to formula (6), we have:

$$\rho_g^D[X] = \frac{1}{1-p}\int_{VaR_p[X]}^{+\infty} (x - E[X])^2 dF_X(x).$$

Let us change the variable in this integral: $x = F_X^{-1}(q) = VaR_q[X]$ aking into account the fact that for q = 1 the variable x takes on the value $+\infty$, and at $q = p$ the variable x takes on the value $VaR_p[X]$. Then we get:

$$\rho_g^D[X] = \frac{1}{1-p}\int_p^1 (VaR_q[X] - E[X])^2 dq =$$

$$= \frac{1}{1-p}\int_p^1 (VaR_q^{rel}[X])^2 dq.$$

The statement is proven.

Note that the value $\tilde{\rho}_g^D[X] = \sqrt{\rho_g^D[X]}$ can also serve as a measure of risk, and its dimension, in contrast to $\rho_g^D[X]$, coincides with the dimension of the random variable X.

It follows from *Statement 2* that this variance distortion risk measure is a new measure of catastrophic risks.

It is of interest to compare the risk estimates obtained using this measure and the risk measure of variance distortion obtained in the previous consideration using the distortion function $g(x) = 1_{\{x>1-p\}},\ p \in (0,1)$.

The following proposition can be proved.

Proposition 1

The following inequality is valid:

$$\rho_g^D[X] \geq (VaR_p[X] - E[X])^2,$$

and hence

$$\tilde{\rho}_g^D[X] \geq |VaR_p[X] - E[X]| = |VaR_p^{rel}[X]|,$$

where through $VaR_p^{rel}[X]$ the relative value of *VaR* is denoted, i.e. the value of the maximum possible unfavorable deviation of a random variable X with a given probability p.

Proof

Formula (6) obviously implies the inequality

$$\rho_g^D[X] \geq \frac{(VaR_p[X] - E[X])^2}{1-p}\int_{VaR_p[X]}^{+\infty} dF_X(x).$$

But

$$\int_{VaR_p[X]}^{+\infty} dF_X(x) = F_X(+\infty) - F_X(F_X^{-1}(p)) = 1 - p.$$

Whence follows the validity of the required inequality:

$$\rho_g^D[X] \geq (VaR_p[X] - E[X])^2,$$

and hence

$$\tilde{\rho}_g^D[X] \geq |VaR_p[X] - E[X]| = |VaR_p^{rel}[X]|.$$

The proposition is proven.

The meaning of this proposition is that this variance distortion risk measure always gives risk estimates that exceed (or equal) the risk estimates obtained using the first proposed measure of the risk of variance distortion corresponding to the distortion function

$$g(x) = 1_{\{x>1-p\}},\ p \in (0,1).$$

COMPOSITE METHOD OF CREATING NEW DISTORTION FUNCTIONS AND VARIANCE DISTORTION RISK MEASURES

The distortion functions can be viewed as a starting point for constructing a family of distortion risk measures. Thus, the construction and selection of distortion functions play an important role in the development of different families of risk measures with different properties. C. Yin, D. Zhu [12] consider three methods: the composite method, mixing methods and copula, which allow constructing new classes of functions and distortion risk measures using the available distortion functions and measures.

In this paper, we will discuss and develop only the first of them — the composite method and apply it to obtain new variance distortion risk measures.

The composite method uses a composition of distortion functions to construct new distortion functions.

Suppose that $h_1, h_2, \ldots$ are distortion functions, we define $f_1(x) = h_1(x)$ and complex functions $f_n(x) = f_{n-1}(h_n(x))$, n = 1, 2, ... It is easy to check that $f_n(x)$, n = 1, 2,... are also distortion functions. If $h_1, h_2, \ldots$ the concave distortion functions, then each $f_n(x)$ is concave and they satisfy the conditions:

$$f_1 \le f_2 \le f_3 \le \ldots$$

We will now construct the distortion functions using the composite method, in the form of a superposition of the known distortion functions, which led to the construction of interesting expectation distortion risk measures. We hope that when applied to the construction of variance distortion risk measures, it will be possible to construct new risk measures with interesting properties.

Examples of variance distortion risk measures obtained using the composite method

Example 1. Let us consider a convex distortion function $g(x) = \frac{e^x - 1}{e - 1}$ and a distortion function obtained as the following superposition of distortion functions:

$$h(x) = 1_{\{x>1-p\}}(g(x)).$$

It is obvious that

$$h(x) = 1_{\{g(x)>1-p\}}(x) = 1_{\{x>\ln(1+(e-1)(1-p))\}}(x) =$$

$$= 1_{\{x>1-(1-\ln(1+(e-1)(1-p)))\}}(x).$$

Then, using *Statement 1*, the last expression yields a formula for the variance distortion risk measure corresponding to a given distortion function $h(x)$:

$$\rho_h^D[X] = (VaR_{1-\ln(1+(e-1)(1-p))}[X] - E[X])^2 =$$

$$= (VaR^{rel}_{1-\ln(1+(e-1)(1-p))}[X])^2,$$

where through $VaR^{rel}_{1-\ln(1+(e-1)(1-p))}[X]$ the corresponding measure of the relative value VaR is denoted. p = 0.95 we obtain $\rho_h^D[X] = (VaR^{rel}_{0.032}[X])^2$.

Example 2. Let us consider a concave distortion function $g(x) = \sin\frac{\pi}{2}x$ and and a distortion function obtained as the following superposition of distortion functions:

$$h(x) = 1_{\{x>1-p\}}(g(x)).$$

It is obvious that

$$h(x) = 1_{\{g(x)>1-p\}}(x) = 1_{\{x>\frac{2}{\pi}\arcsin(1-p)\}}(x) =$$

$$= 1_{\{x>1-(1-\frac{2}{\pi}\arcsin(1-p))\}}(x).$$

Then, using *Statement 1*, the last expression yields a formula for the risk measure of variance distortion corresponding to a given distortion function $h(x)$:

$$\rho_h^D[X] = (VaR_{1-\frac{2}{\pi}\arcsin(1-p)}[X] - E[X])^2 =$$

$$= (VaR^{rel}_{1-\frac{2}{\pi}\arcsin(1-p)}[X])^2,$$

where through $VaR^{rel}_{1-\frac{2}{\pi}\arcsin(1-p)}[X]$ the corresponding measure of the relative value VaR is denoted.

For p = 0.95 we obtain $\rho_h^D[X] = (VaR^{rel}_{0.9682}[X])^2$.

Example 3. Let us consider a concave distortion function $g(x) = \frac{\ln(x+1)}{\ln 2}$ and a distortion function obtained as the following superposition of distortion functions:

$$h(x) = 1_{\{x>1-p\}}(g(x)).$$

It is obvious that

$$h(x) = 1_{\{g(x)>1-p\}}(x) = 1_{\{x>2^{1-p}-1\}}(x) = 1_{\{x>1-(2-2^{1-p})\}}(x).$$

Then, using *Statement 1*, the last expression yields a formula for the risk measure of variance distortion corresponding to a given distortion function $h(x)$:

$$\rho_h^D[X] = (VaR_{2-2^{1-p}}[X] - E[X])^2 = (VaR_{2-2^{1-p}}^{rel}[X])^2,$$

where through $VaR_{2-2^{1-p}}^{rel}[X]$ the corresponding measure of the relative value *VaR* is denoted.

For $p = 0.95$ we obtain $\rho_h^D[X] = (VaR_{0.97}^{rel}[X])^2$.

Example 4. Let us consider a concave distortion function $g(x) = x^{\alpha}, 0 < \alpha < 1$ and a distortion function obtained as the following superposition of distortion functions:

$$h(x) = 1_{\{x>1-p\}}(g(x)).$$

It is obvious that

$$h(x) = 1_{\{g(x)>1-p\}}(x) = 1_{\{x>(1-p)^{\frac{1}{\alpha}}\}}(x) = \\ = 1_{\{x>1-(1-(1-p)^{\frac{1}{\alpha}})\}}(x).$$

Then, using *Statement 1*, the last expression yields a formula for the risk measure of variance distortion corresponding to a given distortion function h(x):

$$\rho_h^D[X] = (VaR_{1-(1-p)^{\frac{1}{\alpha}}}[X] - E[X])^2 = \\ = (VaR^{rel}_{1-(1-p)^{\frac{1}{\alpha}}}[X])^2,$$

where through $VaR^{rel}_{1-(1-p)^{\frac{1}{\alpha}}}[X]$ the corresponding measure of the relative value *VaR* is denoted.

For $\alpha = \frac{1}{2}$ and p = 0.95 we obtain $\rho_h^D[X] = (VaR_{0.9975}^{rel}[X])^2$.

Example 5. Let us consider a concave distortion function $g(x) = xe^{1-x}$ and a distortion function obtained as the following superposition of distortion functions:

$$h(x) = 1_{\{x>1-p\}}(g(x)).$$

It is obvious that

$$h(x) = 1_{\{g(x)>1-p\}}(x) = 1_{\{x>-W(-\frac{1-p}{e})\}}(x) = \\ = 1_{\{x>1-(1+W(-\frac{1-p}{e}))\}}(x),$$

where through $W(x)$ the well-known Lambert function is denoted (for the Lambert function and its properties, see [19], an example of its application, see [16]).

Then, using *Statement 1*, the last expression yields the formula for the variance distortion function corresponding to the given distortion function h(x):

$$\rho_h^D[X] = (VaR_{1+W(-\frac{1-p}{e})}[X] - E[X])^2 = \\ = (VaR^{rel}_{1+W(-\frac{1-p}{e})}[X])^2,$$

where through $VaR^{rel}_{1-W(-\frac{1-p}{e})}[X]$ the corresponding measure of the relative value *VaR* is denoted (see [16]). For $p = 0.95$ we obtain $\rho_h^D[X] = (VaR_{0.9813}^{rel}[X])^2$.

Variance distortion risk measure obtained by superposition of distortion functions:

$$1_{\{x>1-p\}}(x), g(x) = \min\{\frac{x}{1-p}, 1\},$$
$$g_{\alpha}(x) = \min\{\frac{x}{1-\alpha p}, 1\}, p \in [0,1] \text{ and } g_n(x) = x^{\frac{1}{n}}.$$

Let us first study the variance distortion risk measures, which can be obtained using the distortion function h(x), obtained using the following superpositions:

$$h(x) = 1_{\{x>1-p\}}(\underbrace{g(g(\ldots(g(x))}_{n-1-\text{раз}}) = 1_{\{x>1-p\}}(g_n(x)).$$

This concave distortion function is represented as:

$$h(x) = \begin{cases} 1, \text{если } x > (1-p)^n \\ 0, \text{если } 0 \le x \le (1-p)^n \end{cases} = 1_{\{x>(1-p)^n\}}(x).$$

As it was shown in [16], this distortion function, in the class of expectation distortion risk measure, corresponds to the risk measure "*ES* to the power of n", $\rho_h[X] = ES_p^{(n)}[X]$, where n — any natural number.

Let us consider a variance distortion risk measure, which corresponds to a distortion function of a given type.

According to formula (2), we have:

$$\rho_g^D[X] = 2\int_{E[X]}^{+\infty} 1_{\{\bar{F}_X(x)>(1-p)^n\}}(x - E[X])dx + \\ +2\int_{-\infty}^{E[X]} [1_{\{\bar{F}_X(x)>(1-p)^n\}} - 1](x - E[X])dx =$$

$$= 2\int_{E[X]}^{+\infty} 1_{\{F_X(x)\le 1-(1-p)^n\}}(x-E[X])dx +$$

$$+2\int_{-\infty}^{E[X]} [1_{\{F_X(x)\le 1-(1-p)^n\}}-1](x-E[X])dx.$$

Denoting by the F_X^{-1} function, nverse to the distribution function F_X, we obtain:

$$\rho_g^D[X] = 2\int_{E[X]}^{+\infty} 1_{\{x\le F_X^{-1}(1-(1-p)^n)\}}(x-E[X])dx +$$

$$+2\int_{-\infty}^{E[X]} [1_{\{x\le F_X^{-1}(1-(1-p)^n)\}}-1](x-E[X])dx. \quad (12)$$

In the further derivation of the formula for $\rho_g^D[X]$ we will have to consider two cases.

A) Suppose that, $F_X^{-1}(1-(1-p)^n) < E[X]$,i.e. $VaR_p^{(n)}[X] < E[X]$.

In this case, it is obvious that the first integral in formula (12) is equal to zero. And we obtain:

$$\rho_g^D[X] = 2\int_{-\infty}^{E[X]} [1_{\{x\le F_X^{-1}(1-(1-p)^n)\}}-1](x-E[X])dx =$$

$$= -(x-E[X])^2 \Big|_{F_X^{-1}(1-(1-p)^n)=VaR_p^{(n)}[X]}^{E[X]} =$$

$$= (VaR_p^{(n)}[X]-E[X])^2.$$

Let us now consider the second case.

B) Suppose that, $F_X^{-1}(1-(1-p)^n) \ge E[X]$, т.е. $VaR_p^{(n)}[X] \ge E[X]$.

In this case, it is obvious that the second integral in formula (12) is equal to zero. And we obtain:

$$\rho_g^D[X] = 2\int_{E[X]}^{+\infty} 1_{\{x\le F_X^{-1}(1-(1-p)^n)\}}(x-E[X])dx =$$

$$= 2\int_{E[X]}^{F_X^{-1}(1-(1-p)^n)} (x-E[X])dx =$$

$$= (x-E[X])^2 \Big|_{E[X]}^{F_X^{-1}(1-(1-p)^n)=VaR_p^{(n)}[X]} =$$

$$= (VaR_p^{(n)}[X]-E[X])^2.$$

Thus, we have proved the following statement.

Statement 4

The variance distortion risk measure corresponding to the distortion function

$$h(x) = 1_{\{x>1-p\}}(\underbrace{g(g(\ldots(g(x))}_{n-1-\text{times}} = 1_{\{x>1-p\}}(g_n(x)),$$

$p \in (0,1)$, is the risk measure

$$\rho_g^D[X] = (VaR_p^{(n)}[X]-E[X])^2.$$

Note that the value $\tilde{\rho}_g^D[X] = \sqrt{\rho_g^D[X]}$ can also serve as a measure of risk, and its dimension, in contrast to $\rho_g^D[X]$, coincides with the dimension of the random variable *X*.

And, obviously,

$$\tilde{\rho}_g^D[X] = |VaR_p^{(n)}[X]-E[X]| = |VaR_p^{(n)rel}[X]|,$$

where through $VaR_p^{(n)rel}[X]$ the relative value of "*VaR* to the power of *n*" is denoted, i.e. the deviation of the risk measure $VaR_p^{(n)}[X]$ of the random variable *X* from the expected value *X*.

Now let us study the risk measures of variance distortion, which can be obtained using the distortion function *h*(*x*), obtained using the following superpositions:

$$h(x) = 1_{\{x>1-p\}}(\underbrace{g(g(\ldots g}_{k-1-\text{раз}}(g_\alpha(x)\ldots) = 1_{\{x>1-p\}}(g_{k-1}(g_\alpha(x)))$$

This concave distortion function is represented as:

$$h(x) = \begin{cases} 1, if \;\; x > (1-p^k(1-\alpha p) \\ 0, if \;\; 0 \le x \le (1-p)^k(1-\alpha p) \end{cases} =$$
$$= 1_{\{x>(1-p)^k(1-\alpha p)\}}(x).$$

As shown in [16], this distortion function, in the class of expectation distortion risk measures, corresponds to the risk measure "*VaR* to the power of *t*", $\rho_h[X] = VaR_p^{(t)}[X]$ (see also [15]), where *t* — any real number represented in the following form: $t = k + \alpha$, where *k* — a natural number, and α — a real number, and $0 \le \alpha < 1$.

Let us consider a measure of the risk of variance distortion, which corresponds to a distortion function of a given type.

According to formula (2), we have:

$$\rho_g^D[X] = 2\int_{E[X]}^{+\infty} 1_{\{\bar{F}_X(x)>(1-p)^k(1-\alpha p)\}}(x-E[X])dx +$$

$$+2\int_{-\infty}^{E[X]}[1_{\{\bar{F}_X(x)>(1-p)^k(1-\alpha p)\}}-1](x-E[X])dx=$$

$$=2\int_{E[X]}^{+\infty}1_{\{F_X(x)\le 1-(1-p)^k(1-\alpha p)\}}(x-E[X])dx+$$

$$+2\int_{-\infty}^{E[X]}[1_{\{F_X(x)\le 1-(1-p)^k(1-\alpha p)\}}-1](x-E[X])dx.$$

Denoting by the F_X^{-1} function, inverse to the distribution function F_X, we get:

$$\rho_g^D[X]=2\int_{E[X]}^{+\infty}1_{\{x\le F_X^{-1}(1-(1-p)^k(1-\alpha p))\}}(x-E[X])dx+$$

$$+2\int_{-\infty}^{E[X]}[1_{\{x\le F_X^{-1}(1-(1-p)^k(1-\alpha p))\}}-1](x-E[X])dx. \quad (13)$$

In the further derivation of the formula for $\rho_g^D[X]$ we will have to consider two cases.

A) Suppose that

$$F_X^{-1}(1-(1-p)^k(1-\alpha p))<E[X],$$

i.e. $VaR_p^{(t)}[X]<E[X]$.

In this case, it is obvious that the first integral in formula (13) is equal to zero. And we obtain:

$$\rho_g^D[X]=2\int_{-\infty}^{E[X]}[1_{\{x\le F_X^{-1}(1-(1-p)^k(1-\alpha p))\}}-1](x-E[X])dx=$$

$$=-2\int_{F_X^{-1}(1-(1-p)^k(1-\alpha p))}^{E[X]}(x-E[X])dx=$$

$$=-(x-E[X])^2\Big|_{F_X^{-1}(1-(1-p)^k(1-\alpha p))=VaR_p^{(t)}[X]}^{E[X]}=$$

$$=(VaR_p^{(t)}[X]-E[X])^2.$$

Let us now consider the second case.

B) Suppose that

$$F_X^{-1}(1-(1-p)^k(1-\alpha p))\ge E[X],$$

i.e. $VaR_p^{(t)}[X]\ge E[X]$.

In this case, it is obvious that the second integral in formula (13) is equal to zero. And we obtain:

$$\rho_g^D[X]=2\int_{E[X]}^{+\infty}1_{\{x\le F_X^{-1}(1-(1-p)^k(1-\alpha p))\}}(x-E[X])dx=$$

$$=2\int_{E[X]}^{F_X^{-1}(1-(1-p)^k(1-\alpha p))}(x-E[X])dx=$$

$$=(x-E[X])^2\Big|_{E[X]}^{F_X^{-1}(1-(1-p)^k(1-\alpha p))=VaR_p^{(t)}[X]}=$$

$$=(VaR_p^{(t)}[X]-E[X])^2.$$

Thus, we have proved the following statement.

Statement 5

The risk measure of the variance distortion corresponding to the distortion function

$$h(x)=1_{\{x>1-p\}}(\underbrace{g(g(\ldots(g(}_{k-1-\text{times}}g_\alpha(x))=1_{\{x>1-p\}}(g_{k-1}(g_\alpha(x)),$$

$p\in(0,1)$, is the risk measure

$$\rho_g^D[X]=(VaR_p^{(t)}[X]-E[X])^2,$$

for any real number t, represented in the following form: $t=k+\alpha$, where k — a natural number, and α — a real number, and $0\le\alpha<1$.

Note that the value $\tilde{\rho}_g^D[X]=\sqrt{\rho_g^D[X]}$ can also serve as a measure of risk, and its dimension, in contrast to $\rho_g^D[X]$, coincides with the dimension of the random variable *X*.

It is obvious that

$$\tilde{\rho}_g^D[X]=|VaR_p^{(t)}[X]-E[X]|=|VaR_p^{(t)rel}[X]|,$$

where through $VaR_p^{(t)rel}[X]$ the relative value "*VaR* to the power of *t*" is denoted, i.e. the deviation of the risk measure $VaR_p^{(t)}[X]$ of the random variable *X* from the expected value *X*.

Now let us study the variance distortion risk measure, which can be obtained using the distortion function *h*(*x*), obtained using the following superpositions:

$$h(x)=1_{\{x>1-p\}}(\underbrace{g(g(\ldots g}_{n-\text{times}}(x)\ldots)$$

This concave distortion function is represented as:

$$h(x)=\begin{cases}1, if\ \ x>(1-p)^n\\ \dfrac{x}{(1-p)^n}, if\ \ 0\le x\le(1-p)^n\end{cases}$$

As shown in [16], this distortion function, in the class of expectation distortion risk measures, corresponds to the risk measure "*VaR* to the power of *n*", $\rho_h[X]=VaR_p^{(n)}[X]$ (see also [16]), where *n* — any natural number.

Let us consider a variance distortion risk measure, which corresponds to a distortion function of a given type.

We note that

$$h(\bar{F}(x))=\begin{cases}1, if\ \bar{F}_X(x)>(1-p)^n\\ \dfrac{\bar{F}_X(x)}{(1-p)^n}, if\ 0\le \bar{F}_X(x)\le(1-p)^n,\end{cases}$$

or

$$h(\bar{F}_X(x))=\begin{cases}1, if\ F_X(x)\le 1-(1-p)^n\\ \dfrac{1-F_X(x)}{(1-p)^n}, if\ F_X(x)>1-(1-p)^n,\end{cases}$$

which means

$$h(\bar{F}_X(x))=\begin{cases}1, if\ x\le F_X^{-1}(1-(1-p)^n)\\ \dfrac{1-F_X(x)}{(1-p)^n}, if\ x> F_X^{-1}(1-(1-p)^n).\end{cases}$$

In the further derivation of the formula for $\rho_g^D[X]$ we will have to consider two cases.

A) Suppose that, $F_X^{-1}(1-(1-p)^n)<E[X]$, i.e. $VaR_p^{(n)}[X]<E[X]$.

Then, according to (2), we obtain:

$$\rho_g^D[X]=2\int_{E[X]}^{+\infty}\frac{1-F_X(x)}{(1-p)^n}(x-E[X])dx+$$

$$+\ 2\int_{F_X^{-1}(1-(1-p)^n)}^{E[X]}[\frac{1-F_X(x)}{(1-p)^n}-1](x-E[X])dx\ =$$

$$=\ 2\int_{F_X^{-1}(1-(1-p)^n)}^{+\infty}\frac{1-F_X(x)}{(1-p)^n}(x-E[X])dx\ -$$

$$-\ 2\int_{F_X^{-1}(1-(1-p)^n)}^{E[X]}(x-E[X])dx=$$

$$=\frac{1}{(1-p)^n}\int_{F_X^{-1}(1-(1-p)^n)}^{+\infty}(1-F_X(x))d(x-E[X])^2-$$

$$-(x-E[X])^2\,|_{F_X^{-1}(1-(1-p)^n)}^{E[X]}.$$

By applying integration by parts in this expression, we obtain:

$$\rho_g^D[X]=\frac{1}{(1-p)^n}(x-E[X])^2(1-F_X(x))|_{F_X^{-1}(1-(1-p)^n)}^{+\infty}+$$

$$+\frac{1}{(1-p)^n}\int_{F_X^{-1}(1-(1-p)^n)}^{+\infty}(x-E[X])^2dF_X(x)+$$

$$+(F_X^{-1}(1-(1-p)^n)-E[X])^2.$$

Then, using assumption A) about the distribution function, we obtain:

$$\rho_g^D[X]=$$

$$=-\frac{1}{(1-p)^n}(F_X^{-1}(1-(1-p)^n)-E[X])^2(1-F_X(F_X^{-1}(1-(1-p)^n)))+$$

$$+\ (F_X^{-1}(1-(1-p)^n)-E[X])^2+$$

$$+\frac{1}{(1-p)^n}\int_{F_X^{-1}(1-(1-p)^n)}^{+\infty}(x-E[X])^2dF_X(x).$$

Then using that

$F_X(F_X^{-1}(1-(1-p)^n)))=1-(1-p)^n$, we obtain

$$\rho_g^D[X]=\frac{1}{(1-p)^n}\int_{F_X^{-1}(1-(1-p)^n)}^{+\infty}(x-E[X])^2dF_X(x)=$$

$$=\frac{1}{(1-p)^n}\int_{VaR_p^{(n)}[X]}^{+\infty}(x-E[X])^2dF_X(x).$$

Let us now consider the second case.

B) Suppose that, $F_X^{-1}(1-(1-p)^n)\ge E[X]$, т.е. $VaR_p^{(n)}[X]\ge E[X]$.

In this case, it is obvious that the second integral in formula (2) is equal to zero. And we obtain:

$$\rho_g^D[X]=2\int_{E[X]}^{+\infty}h(\bar{F}_X(x))(x-E[X])dx=$$

$$=2\int_{E[X]}^{F_X^{-1}(1-(1-p)^n)}(x-E[X])dx+$$

$$+2\int_{F_X^{-1}(1-(1-p)^n)}^{+\infty}\frac{1-F_X(x)}{(1-p)^n}(x-E[X])dx\ =$$

$$(x-E[X])^2\,|_{E[X]}^{F_X^{-1}(1-(1-p)^n)=VaR_p^{(n)}[X]}+$$

$$+\frac{1}{(1-p)^n}\int_{F_X^{-1}(1-(1-p)^n)}^{+\infty}(1-F_X(x))d(x-E[X])^2.$$

By integrating the integral in this expression by parts, we get:

$$\rho_g^D[X] = (F_X^{-1}(1-(1-p)^n) - E[X])^2 +$$

$$+\frac{1}{(1-p)^n}(x-E[X])^2(1-F_X(x))\Big|_{F_X^{-1}(1-(1-p)^n)}^{+\infty} +$$

$$+\frac{1}{(1-p)^n}\int_{F_X^{-1}(1-(1-p)^n)}^{+\infty}(x-E[X])^2 dF_X(x).$$

Then, using assumption A) about the distribution function, we get:

$$\rho_g^D[X] = (F_X^{-1}(1-(1-p)^n) - E[X])^2 -$$

$$-\frac{1}{(1-p)^n}(F_X^{-1}(1-(1-p)^n) - E[X])^2(1-F_X(F_X^{-1}(1-(1-p)^n)))+$$

$$+\frac{1}{(1-p)^n}\int_{F_X^{-1}(1-(1-p)^n)}^{+\infty}(x-E[X])^2 dF_X(x).$$

Then using that $F_X(F_X^{-1}(1-(1-p)^n)) = 1-(1-p)^n$, we obtain

$$\rho_g^D[X] = \frac{1}{(1-p)^n}\int_{F_X^{-1}(1-(1-p)^n)}^{+\infty}(x-E[X])^2 dF_X(x) = \frac{1}{(1-p)^n}\int_{VaR_p^{(n)}[X]}^{+\infty}(x-E[X])^2 dF_X(x). \quad (14)$$

Remembering the variance formula:

$$D[X] = E[(X-E[X])^2] = \int_{-\infty}^{+\infty}(x-E[X])^2 dF_X(x)$$

and comparing it with formula (14), and also taking into account that $P\{X > VaR_p^{(n)}[X]\} = (1-p)^n$, we obtain the following representation for this risk measure:

$$\rho_g^D[X] = E[(X-E[X])^2 \mid X > VaR_p^{(n)}[X]], \quad (15)$$

Or, if you define the conditional variance, provided $X > VaR_p^{(n)}[X]$ by the expression: $D[X \mid X > VaR_p^{(n)}[X]] = E[(X-E[X])^2 \mid X > VaR_p^{(n)}[X]]$, we obtain the following representation for this variance distortion risk measures:

$$\rho_g^D[X] = D[X \mid X > VaR_p^{(n)}[X]]. \quad (16)$$

Thus, we have proved the following statement.

Statement 6

the variance distortion risk measure corresponding to the distortion function $h(x) = \underbrace{g(g(\ldots(g(x))}_{n-\text{раз}}$, $p \in (0,1)$, is the risk measure

$$\rho_g^D[X] = \frac{1}{(1-p)^n}\int_{VaR_p^{(n)}[X]}^{+\infty}(x-E[X])^2 dF_X(x);$$

this risk measure can also be represented as

$$\rho_g^D[X] = D[X \mid X > VaR_p^{(n)}[X]],$$

where

$$D[X \mid X > VaR_p^{(n)}[X]] = E[(X-E[X])^2 \mid X > VaR_p^{(n)}[X]].$$

That is, this risk measure of losses represents the conditional variance of the random factor X, which represents a risk, provided that the value of these losses exceeded the value $VaR_p^{(n)}[X]$.

In [16] the risk measure "*ES* to the power of *n*" was introduced, which turned out (see [16]) to be an expectation distortion risk measure, which we will denote as $ES_p^{(n)}[X]$. It represents the magnitude of the expected tail losses exceeding $VaR_p^{(n)}[X]$, i.e. by definition

$$ES_p^{(n)}[X] = E[X \mid X > VaR_p^{(n)}[X]]. \quad (17)$$

Hence, assuming the continuity of the distribution of losses, the following useful representation was obtained for $ES_p^{(n)}[X]$:

$$ES_p^{(n)}[X] = \frac{1}{(1-p)^n}\int_{[1-(1-p)^n,1]} VaR_q[X]dq. \quad (18)$$

Comparing formula (17) and the presentation of our new risk measure for variance distortion $\rho_g^D[X]$ in section b) *Statement 6*, we see that in the class of risk measures for variance distortion, the new risk measure $\rho_g^D[X]$ has the same significance as the measure $ES_p^{(n)}[X]$ as the measure in the class of expectation distortion risk measures.

Hence, we can conclude that the significance of this measure for the theory and practice of risk management is not less than the significance of risk measures $ES_p^{(n)}[X]$.

Also, looking at formula (18), I would like to have a formula for our variance distortion risk measure $\rho_g^D[X]$ in a form similar to formula (17) for the risk measure *ES*.

The following proposition can be proved.

Statement 7

The variance distortion risk measure $\rho_g^D[X]$, corresponding to the distortion function

$$h(x) = 1_{\{x>1-p\}}(\underbrace{g(g(\ldots g}_{n-\text{раз}}(x)\ldots),$$

where $g(x) = \min\{\frac{x}{1-p}, 1\}$, $p \in [0,1]$, can be represented as:

$$\rho_g[X] = \frac{1}{(1-p)^n}\int_{[1-(1-p)^n,1]} VaR_q^{(n)rel}[X]dq, \quad (19)$$

where

$VaR_q^{(n)rel}[X] = VaR_q^{(n)}[X] - E[X]$, the value of the corresponding relative risk measure $VaR_q^{(n)}[X]$.

Proof

According to formula (6), we have:

$$\rho_g^D[X] = \frac{1}{(1-p)^n} \int_{VaR_p^{(n)}[X]}^{+\infty} (x - E[X])^2 dF_X(x).$$

Let us change the variable in this integral: $x = F_X^{-1}(1-(1-q)^n) = VaR_q^{(n)}[X]$ taking into account the fact that for $q = 1$ the variable x takes on the value $+\infty$, and for $q = p$ the variable x takes on the value $VaR_p^{(n)}[X]$. Than we obtain:

$$\rho_g^D[X] = \frac{1}{(1-p)^n} \int_{1-(1-p)^n}^{1} (VaR_q^{(n)}[X] - E[X])^2 dq =$$

$$= \frac{1}{(1-p)^n} \int_{[1-(1-p)^n,1]} VaR_q^{(n)rel}[X] dq .$$

The statement is proven.

Note that the value $\tilde{\rho}_g^D[X] = \sqrt{\rho_g^D[X]}$ can also serve as a measure of risk, and its dimension, in contrast to $\rho_g^D[X]$, coincides with the dimension of the random variable X.

It follows from *Statement 6* that this variance distortion risk measure represents a new measure of catastrophic risks.

It is of interest to compare the risk estimates obtained using this measure and the variance distortion risk measure obtained in the previous consideration using the distortion functions of the form $h(x) = 1_{\{x>1-p\}}(\underbrace{g(g(\ldots g}_{n-1-\text{times}}(x)\ldots) = 1_{\{x>1-p\}}(g_n(x))$.

The following proposition can be proved.

Proposition 2

The following inequality is valid:

$$\rho_g^D[X] \ge (VaR_p^{(n)}[X] - E[X])^2 ,$$

and hence

$$\tilde{\rho}_g^D[X] \ge |VaR_p^{(n)}[X] - E[X]| = |VaR_p^{(n)rel}[X]| ,$$

where through $VaR_p^{(n)rel}[X]$ a relative value $VaR_p^{(n)}[X]$, is denoted, i.e. the value of the deviation of the risk measure $VaR_p^{(n)}[X]$ of the random variable X from its expected value.

Proof

Formula (13) obviously implies the inequality

$$\rho_g^D[X] \ge \frac{(VaR_p^{(n)}[X] - E[X])^2}{(1-p)^n} \int_{VaR_p^{(n)}[X]}^{+\infty} dF_X(x).$$

But

$$\int_{VaR_p^{(n)}[X]}^{+\infty} dF_X(x) = F_X(+\infty) - F_X(F_X^{-1}(1-(1-p)^n) = (1-p)^n.$$

Whence follows the validity of the required inequality:

$$\rho_g^D[X] \ge (VaR_p^{(n)}[X] - E[X])^2 ,$$

and hence

$$\tilde{\rho}_g^D[X] \ge |VaR_p^{(n)}[X] - E[X]| = |VaR_p^{(n)rel}[X]| .$$

The proposition is proven.

The meaning of this proposition is that this variance distortion risk measure always gives risk estimates that exceed (or equal) the risk estimates obtained using the first proposed variance distortion risk measure corresponding to the distortion function

$$h(x) = 1_{\{x>1-p\}}(\underbrace{g(g(\ldots g}_{n-1-\text{times}}(x)\ldots) = 1_{\{x>1-p\}}(g_n(x))$$

$$p \in (0,1) .$$

Now let us study the variance distortion risk measure, which can be obtained using the distortion function $h(x)$, obtained in the form of any superposition of functions $g(x) = \min\{\frac{x}{1-p}, 1\}$ and a function $g_\alpha(x) = \min\{\frac{x}{1-\alpha p}, 1\}$ of the following form: $h(x) = \underbrace{g(g(\ldots(g(}_{k-\text{times}} g_\alpha(x))\ldots)$. For any real $t \ge 1$, represented in the from where $t = k + \alpha$, where k — a natural number, and α — a real number, $0 < \alpha < 1$, in the class of the expectation distortion risk measure, this distortion function corresponds to theexpectation distortion risk measure "*ES* to the power of *t*", i.e. $ES_p^{(t)}[X]=\rho_h[X]$ (see [9]).

Let us examine the question: what functions of the variance distortion correspond to the given distortion functions *h*?

We note that

$$h(\bar{F}_X(x))=\begin{cases}1, if\ \bar{F}_X(x)>(1-p)^k(1-\alpha p)\\ \dfrac{\bar{F}_X(x)}{(1-p)^k(1-\alpha p)},\\ if\ 0\le \bar{F}_X(x)\le(1-p)^k(1-\alpha p),\end{cases}$$

or

$$h(\bar{F}_X(x))=\begin{cases}1, if\ F_X(x)\le 1-(1-p)^k(1-\alpha p)\\ \dfrac{1-F_X(x)}{(1-p)^k(1-\alpha p)},\\ if\ F_X(x)>1-(1-p)^k(1-\alpha p),\end{cases}$$

Which means

$$h(\bar{F}_X(x))=\begin{cases}1, if\ x\le F_X^{-1}(1-(1-p)^k(1-\alpha p))\\ \dfrac{1-F_X(x)}{(1-p)^k(1-\alpha p)},\\ if\ x> F_X^{-1}(1-(1-p)^k(1-\alpha p)),\end{cases}$$

In the further derivation of the formula for $\rho_g^D[X]$ we will have to consider two cases.

A) Suppose that

$$F_X^{-1}(1-(1-p)^k(1-\alpha p))<E[X],$$

i.e. $VaR_p^{(t)}[X]<E[X]$.

Then, according to (2), we obtain:

$$\rho_g^D[X]=2\int_{E[X]}^{+\infty}\frac{1-F_X(x)}{(1-p)^k(1-\alpha p)}(x-E[X])dx+$$

$$+2\int_{F_X^{-1}(1-(1-p)^k(1-\alpha p))}^{E[X]}\left[\frac{1-F_X(x)}{(1-p)^k(1-\alpha p)}-1\right](x-E[X])dx=$$

$$=2\int_{F_X^{-1}(1-(1-p)^k(1-\alpha p))}^{+\infty}\frac{1-F_X(x)}{(1-p)^k(1-\alpha p)}(x-E[X])dx-$$

$$-2\int_{F_X^{-1}(1-(1-p)^k(1-\alpha p))}^{E[X]}(x-E[X])dx=$$

$$=\frac{1}{(1-p)^k(1-\alpha p)}\int_{F_X^{-1}(1-(1-p)^k(1-\alpha p))}^{+\infty}(1-F_X(x))d(x-E[X])^2-$$

$$-(x-E[X])^2\Big|_{F_X^{-1}(1-(1-p)^k(1-\alpha p))}^{E[X]}.$$

By applying integration by parts in this expression, we obtain:

$$\rho_g^D[X]=$$

$$=\frac{1}{(1-p)^k(1-\alpha p)}(x-E[X])^2(1-F_X(x))\Big|_{F_X^{-1}(1-(1-p)^k(1-\alpha p))}^{+\infty}+$$

$$+\frac{1}{(1-p)^k(1-\alpha p)}\int_{F_X^{-1}(1-(1-p)^k(1-\alpha p))}^{+\infty}(x-E[X])^2dF_X(x)+$$

$$+(F_X^{-1}(1-(1-p)^k(1-\alpha p))-E[X])^2.$$

Then, using assumption A) about the distribution function, we get:

$$\rho_g^D[X]=$$

$$=-\frac{1}{(1-p)^k(1-\alpha p)}(F_X^{-1}(1-(1-p)^k(1-\alpha p))-$$

$$-E[X])^2(1-F_X(F_X^{-1}(1-(1-p)^k(1-\alpha p))))+$$

$$+(F_X^{-1}(1-(1-p)^k(1-\alpha p))-E[X])^2+$$

$$+\frac{1}{(1-p)^k(1-\alpha p)}\int_{F_X^{-1}(1-(1-p)^k(1-\alpha p))}^{+\infty}(x-E[X])^2dF_X(x).$$

And then, using that

$$F_X(F_X^{-1}(1-(1-p)^k(1-\alpha p))))=1-(1-p)^k(1-\alpha p),$$

we get

$$\rho_g^D[X]=$$

$$\frac{1}{(1-p)^k(1-\alpha p)}\int_{F_X^{-1}(1-(1-p)^k(1-\alpha p))}^{+\infty}(x-E[X])^2dF_X(x)=$$

$$=\frac{1}{(1-p)^k(1-\alpha p)}\int_{VaR_p^{(t)}[X]}^{+\infty}(x-E[X])^2dF_X(x).$$

Let us now consider the second case.

B) Suppose that

$$F_X^{-1}(1-(1-p)^k(1-\alpha p)) \geq E[X],$$

i.e. $VaR_p^{(t)}[X] \geq E[X]$.

In this case, it is obvious that the second integral in formula (2) is equal to zero. And we get:

$$\rho_g^D[X] = 2\int_{E[X]}^{+\infty} h(\bar{F}_X(x))(x-E[X])dx =$$

$$= 2\int_{E[X]}^{F_X^{-1}(1-(1-p)^k(1-\alpha p))} (x-E[X])dx +$$

$$+2\int_{F_X^{-1}(1-(1-p)^k(1-\alpha p))}^{+\infty} \frac{1-F_X(x)}{(1-p)^k(1-\alpha p)}(x-E[X])dx =$$

$$=(x-E[X])^2 \Big|_{E[X]}^{F_X^{-1}(1-(1-p)^k(1-\alpha p))=VaR_p^{(t)}[X]} +$$

$$+\frac{1}{(1-p)^k(1-\alpha p)}\int_{F_X^{-1}(1-(1-p)^k(1-\alpha p))}^{+\infty} (1-F_X(x))d(x-E[X])^2.$$

By integrating the integral in this expression by parts, we obtain:

$$\rho_g^D[X] = (F_X^{-1}(1-(1-p)^k(1-\alpha p))-E[X])^2 +$$

$$+\frac{1}{(1-p)^k(1-\alpha p)}(x-E[X])^2(1-F_X(x))\Big|_{F_X^{-1}(1-(1-p)^k(1-\alpha p))}^{+\infty} +$$

$$+\frac{1}{(1-p)^k(1-\alpha p)}\int_{F_X^{-1}(1-(1-p)^k(1-\alpha p))}^{+\infty} (x-E[X])^2 dF_X(x).$$

Then, using assumption A) about the distribution function, we get:

$$\rho_g^D[X] = (F_X^{-1}(1-(1-p)^k(1-\alpha p))-E[X])^2 -$$

$$-\frac{1}{(1-p)^k(1-\alpha p)}(F_X^{-1}(1-(1-p)^k(1-\alpha p))-$$

$$-E[X])^2(1-F_X(F_X^{-1}(1-(1-p)^k(1-\alpha p))))+$$

$$+\frac{1}{(1-p)^k(1-\alpha p)}\int_{F_X^{-1}(1-(1-p)^k(1-\alpha p))}^{+\infty} (x-E[X])^2 dF_X(x).$$

And then using, that

$$F_X(F_X^{-1}(1-(1-p)^k(1-\alpha p))) = 1-(1-p)^k(1-\alpha p),$$

we obtain

$$\rho_g^D[X] =$$

$$\frac{1}{(1-p)^k(1-\alpha p)}\int_{F_X^{-1}(1-(1-p)^k(1-\alpha p))}^{+\infty} (x-E[X])^2 dF_X(x).$$

T.e.

$$\rho_g^D[X] = \frac{1}{(1-p)^k(1-\alpha p)}\int_{VaR_p^{(t)}[X]}^{+\infty} (x-E[X])^2 dF_X(x). \quad (20)$$

Remembering the variance formula:

$$D[X] = E[(X-E[X])^2] = \int_{-\infty}^{+\infty} (x-E[X])^2 dF_X(x)$$

and comparing it with formula (20), and also considering that $P\{X > VaR_p^{(t)}[X]\} = (1-p)^k(1-\alpha p)$, we obtain the following representation for this risk measure:

$$\rho_g^D[X] = E[(X-E[X])^2 \mid X > VaR_p^{(t)}[X]], \quad (21)$$

Or, if you define the conditional variance, provided $X > VaR_p^{(t)}[X]$ by the expression:

$$D[X \mid X > VaR_p^{(t)}[X]] = E[(X-E[X])^2 \mid X > VaR_p^{(t)}[X]],$$

we obtain the following representation for this variance distortion risk measures:

$$\rho_g^D[X] = D[X \mid X > VaR_p^{(t)}[X]]. \quad (22)$$

Thus, we have proved the following statement.

Statement 8

the variance distortion risk measure corresponding to the distortion function

$h(x) = \underbrace{g(g(\ldots(g(}_{k-\text{times}} g_\alpha(x))\ldots)$, is the risk measure

$$\rho_g^D[X] = \frac{1}{(1-p)^k(1-\alpha p)}\int_{VaR_p^{(t)}[X]}^{+\infty} (x-E[X])^2 dF_X(x).$$

this risk measure can also be represented as

$$\rho_g^D[X] = D[X \mid X > VaR_p^{(t)}[X]],$$

where

$$D[X \mid X > VaR_p^{(t)}[X]] = E[(X-E[X])^2 \mid X > VaR_p^{(t)}[X]].$$

That is, this risk measure of losses represents the conditional variance of the random factor X, which represents a risk, provided that the value of these losses exceeded the value $VaR_p^{(t)}[X]$.

In [16], the risk measure "ES to the power of t" was introduced, which turned out (see [16]) to be an expectation distortion risk measure, which we will denote as $ES_p^{(t)}[X]$. It represents the magnitude of the expected tail losses, exceeding $VaR_p^{(t)}[X]$, i.e. by definition

$$ES_p^{(t)}[X] = E[X \mid X > VaR_p^{(t)}[X]]. \qquad (23)$$

Hence, under the assumption of continuity of the distribution of losses, the following useful representation was obtained for $ES_p^{(n)}[X]$:

$$ES_p^{(t)}[X] = \frac{1}{(1-p)^k(1-\alpha p)} \int\limits_{[1-(1-p)^k(1-\alpha p),1]} VaR_q[X]dq. \quad (24)$$

Comparing formula (23) and the presentation of our new variance distortion risk measure $\rho_g^D[X]$ in section b) of *Statement 8*, we see that in the class of variance distortion risk measures, the new risk measure $\rho_g^D[X]$ has the same significance as the measure $ES_p^{(t)}[X]$ in the class of expectation distortion risk measures.

Hence, we can conclude that the significance of this measure for the theory and practice of risk management is not less than the significance of risk measures $ES_p^{(t)}[X]$.

Looking at formula (24), I would like to have a formula for our variance distortion risk measure $\rho_g^D[X]$ in a form similar to formula (23) for the risk measure ES.

The following proposition can be proved.

Statement 9

The variance distortion risk measure $\rho_g^D[X]$, corresponding to the distortion function

$h(x) = \underbrace{g(g(\ldots(g(}_{k-\text{times}} g_\alpha(x))\ldots)$, can be represented as:

$$\rho_g[X] = \frac{1}{(1-p)^k(1-\alpha p)} \int\limits_{[1-(1-p)^k(1-\alpha p),1]} VaR_q^{(t)rel}[X]dq, \quad (25)$$

where

$VaR_q^{(t)rel}[X] = VaR_q^{(t)}[X] - E[X]$, the value of the corresponding relative risk measure $VaR_q^{(t)}[X]$.

Proof

According to formula (20), we have:

$$\rho_g^D[X] = \frac{1}{(1-p)^k(1-\alpha p)} \int\limits_{VaR_p^{(t)}[X]}^{+\infty} (x - E[X])^2 dF_X(x).$$

Let us change the variable in this integral: $x = F_X^{-1}(1-(1-q)^k(1-\alpha q)) = VaR_q^{(t)}[X]$ taking into account the fact that for $q = 1$ the variable x takes on the value $+\infty$, and for $q = p$ the variable x takes on the value $VaR_p^{(t)}[X]$. Then we obtain:

$$\rho_g^D[X] = \frac{1}{(1-p)^k(1-\alpha p)} \int\limits_{1-(1-p)^k(1-\alpha p)}^{1} (VaR_q^{(t)}[X] - E[X])^2 dq =$$

$$= \frac{1}{(1-p)^k(1-\alpha p)} \int\limits_{[1-(1-p)^k(1-\alpha p),1]} VaR_q^{(t)rel}[X]dq.$$

The statement is proven.

Note that the value $\tilde{\rho}_g^D[X] = \sqrt{\rho_g^D[X]}$ can also serve as a measure of risk, and its dimension, in contrast to $\rho_g^D[X]$, coincides with the dimension of the random variable X.

It follows from *Statement 8* that this variance distortion risk measure represents a new measure of catastrophic risks.

It is of interest to compare the risk estimates obtained using this measure and the risk measure of variance distortion obtained in the previous consideration using the distortion functions of the form $h(x) = 1_{\{x>1-p\}}(\underbrace{g(g(\ldots g}_{k-1-\text{times}}(g_\alpha(x)\ldots) = 1_{\{x>1-p\}}(g_{k-1}(g_\alpha(x)))$.

The following proposition can be proved.

Proposition 3

The following inequality is valid:

$$\rho_g^D[X] \geq (VaR_p^{(t)}[X] - E[X])^2,$$

and hence

$$\tilde{\rho}_g^D[X] \geq |VaR_p^{(t)}[X] - E[X]| = |VaR_p^{(t)rel}[X]|,$$

where through $VaR_p^{(t)rel}[X]$ a relative value $VaR_p^{(t)}[X]$, is denoted, the value of the deviation of the risk measure $VaR_p^{(t)}[X]$ of the random variable X from its expected value.

Proof

From formula (20), obviously, the inequality follows:

$$\rho_g^D[X] \geq \frac{(VaR_p^{(t)}[X]-E[X])^2}{(1-p)^k(1-\alpha p)} \int_{VaR_p^{(t)}[X]}^{+\infty} dF_X(x).$$

But

$$\int_{VaR_p^{(t)}[X]}^{+\infty} dF_X(x) = F_X(+\infty) -$$

$$-F_X(F_X^{-1}(1-(1-p)^k(1-\alpha p)) = (1-p)^k(1-\alpha p).$$

Whence follows the validity of the required inequality:

$$\rho_g^D[X] \geq (VaR_p^{(t)}[X]-E[X])^2,$$

and hence

$$\tilde{\rho}_g^D[X] \geq |VaR_p^{(t)}[X]-E[X]| = |VaR_p^{(t)rel}[X]|.$$

The proposition is proven.

The meaning of this proposition is that this variance distortion risk measure always gives risk estimates that exceed (or equal) the risk estimates obtained using the first proposed variance distortion risk measure corresponding to the distortion function

$$h(x) = 1_{\{x>1-p\}}(\underbrace{g(g(\ldots g}_{k-1-\text{times}}(g_\alpha(x)\ldots) = 1_{\{x>1-p\}}(g_{k-1}(g_\alpha(x))),$$

$p \in (0,1)$, i.e. risk measures $(VaR_p^{(t)}[X]-E[X])^2$.

CONCLUSIONS

In the last decade, there has been a vigorous theoretical study of a class of risk measures called distortion risk measures, and they have become widely used in financial and insurance applications due to their attractive properties. This paper introduces a new concept of variance distortion risk measures and explores some of their properties. A large number of examples of variance distortion risk measures are considered and the possibility of their application for assessing risks of various degrees of catastrophicity is investigated. In this paper, the authors introduced and investigated the risk measures "*VaR* to the power of *t*" and risk measures "*ES* to the power of *t*" into scientific circulation. In them, using the composite method, it was proved that these measures also belong to the class of the expectation distortion risk measures, and the corresponding distortion functions are described. In this paper, we search for the variance distortion risk measures using the same distortion functions that were used to construct risk measures "*VaR* to the power of *t*" and "*ES* to the power of *t*" as subsets of the expectation distortion risk measures. At the same time, such variance distortion risk measures were identified as the square of the relative value of the risk measure "*VaR* to the power of *t*" $\rho_g^D[X] = (VaR_p^{(t)}[X]-E[X])^2$ and the risk measure, which represents the conditional variance of the random factor X provided that the value of these losses exceeded the value $VaR_p^{(t)}[X]$, and various formulas were obtained to represent these variance distortion risk measures. The paper investigates the question of the relationship between these variance distortion risk measures.

The expectation distortion risk measures are currently well studied and have many useful and convenient properties. This paper opens up an interesting area of research in assessing variance distortion risks. It seems interesting both to study the general properties of variance distortion risk measures and to find new variance distortion risk measures with special properties that make it possible to identify financial risks of varying degrees of catastrophicity.

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ABOUT THE AUTHOR

Vigen B. Minasyan — Cand. Sci. (Phys.-Math.), Assoc. Prof., Head of Limitovskii Corporate Finance, Investment Design and Evaluation Department, Higher School of Finance and Management, Russian Presidential Academy of National Economy and Public Administration, Moscow, Russia

minasyanvb@ranepa.ru, minasyanvb@yandex.ru

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