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Upper Limits of Financial Risk Measures of Various Degrees of Catastrophicity

V.B. Minasyan
Higher School of Finance and Management, Russian Presidential Academy of National Economy and Public Administration, Moscow, Russia

ABSTRACT

The question of assessing the magnitude of risks using certain risk measures presents one of the most important problems of modern finance. However, many modern risk measures require considerable effort at times and, in practice, the investor would have sufficient knowledge of the upper limits of those risks. Comparing them with their risk appetite, an investor, in the case when the upper limits of risk measures would fit into their risk appetite, could assess this risk as acceptable to themselves. Only if the upper limit of the appropriate risk measure exceeded their risk appetite would there be a need for a detailed assessment of the appropriate risk measure. **The aim** of this paper is to consider upper limits first for known risk measures such as value at risk, *VaR*, and expected deficit or notional value at risk of *ES*. Next, upper limits are obtained for the risk measures *VaR* to the degree of t, $VaR^{(t)}$ and ES to the degree of t, $ES^{(t)}$ introduced by the author into scientific use. Also, using the results of V. Hürlimann, representations for maximum values of risk measures $VaR^{(t)}$ and $ES^{(t)}$. The **method** of obtaining the described results is the application of certain representations of all these risk measures, the application of P. Chebyshev's inequalities, as well as the results of V. Hürlimann. As a **result** of the study, descriptions have been proposed for the upper limits, expressing them only after a few first moments of the loss distribution law. The author **concludes** that the study of the upper limits of important risk measures of scientific interest has practical value for the express assessment of relevant risks.

Keywords: upper limits of risk measures; *VaR* risk measure; *ES* risk measure; $VaR^{(t)}$ risk measures; $ES^{(t)}$ risk measures; catastrophic risk measures

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INTRODUCTION

One of the central issues of risk management is the idea of *extremum* of certain risk measures assessed in relation to the most important risks for it. But in reality, a risk manager sometimes acts conservatively, based on the least attractive risk measure, which is consistent with the incomplete information available to it. This can be done by identifying an upper limit for the measure in a given risk class consistent with partially known information [1].

In this article we examine the upper limits of the risks assessed by certain risk measures distorted expectations, when the main risk is not fully defined and only some information about its timing is available. This problem is relevant for various reasons. First, risk measures of distorted expectation have many important properties that are generally expected to have "good" risk measures [2]. Second, portfolio risk measurement is at the center of risk management. When the margin functions of the allocation of portfolios as well as the asset dependency structure are known, portfolio risk can be quantified using, for example, Monte Carlo simulation. In most cases, however, full information on the dependency structure is not expected to be available, and various stakeholders, such as investors and regulators, may be interested in learning the worst-case scenario for the portfolio (i.e. scenarios where the risk measure reaches the highest). In this regard, note that there is a rich literature on the search for limits for the quantiles — also called value at risk (*VaR*) portfolio, provided that all margin distribution functions are known, but the dependencies are unknown [3–8].

In this article, however, we do not fix the marginal distribution functions, but we get limits when we only know some moments of portfolio loss (for example, based on portfolio statistics) without specifying the marginal distribution functions.

The most famous risk measures of distorted expectation are the *VaR* value and the notional value at risk, also called as expected shortfall *ES* measure in the literature [1]. Actually, *ES* is the smallest coherent risk measure that is more *VaR* risk measure, which is the most commonly used risk measure in risk management and market surveillance practices, but is not a subadditive and therefore a coherent risk measure [2]. In fact, *VaR* is a specific distribution quantum, while ES is more focused on the right end of the distribution in the sense that it measures the expected loss provided it is greater than *Va* R. Instant limits for *VaR* and *ES* have been studied in literature by several authors including Kaas, Goovaerts [9], Denuit etc. [10], De Schepper, Heijnen [11], Hürlimann [12, 13]. In particular, Hürlimann [12] finds analytical limits for *VaR* and *ES* with knowledge of medium, variance, asymmetry and excess.

In this context, it should be noted that it cannot be expect, that there is a risk measure (i.e. one number) that describes all risk characteristics and provides a complete picture of portfolio risk (i.e. a random value). For example, Hürlimann research [12] about *ES* for various two-parameter distribution functions with fixed mean and loss probability variance show that *ES* does not always correctly reflect the increase in risk (tail) from one distribution to another. Moreover, risk measures are used in different contexts such as risk management (McNeil et al. [14]), price formation (Wirch, Hardy [15]), capital allocation (Dhaene et al. [16]) and regulation (Danielsson et al. [17]), and risk management suitable for one purpose, could be inappropriate in another context.

V. B. Minasyan [18] introduced *VaR* risk measures in the degree t, and [19] proved that the *VaR* measures in the degree t is a subset of the risk measures of distorted expectation. That is, every *VaR* risk measure in the degree t ($VaR_p^{(t)}$) at any $t \geqslant 1$ is a risk measures of distorted expectation with a certain distortion function. The function was presented. In the latest paper, a group of new risk measures called "*ES* in the degree t" ($ES_p^{(t)}$) at any confidence probability p and any real $t \geqslant 1$ were introduced. The paper examined the relationship between two classes of risk measures: risk measures of distorted expectation and *ES* measures in the degree t, and proved that the group of *ES* measures in the degree t is a subset of the set of risk measures of distorted expectation. That is, that every *ES* risk measure in the degree t at any $t \geqslant 1$ is risk measures of distorted expectation with a certain distortion function. The function was presented.

As mentioned, there cannot be a separate risk measure, which is able to cover all risk characteristics. There is no such ideal measure. The group of *VaR* risk measures in the degree t and *ES* in the degree t, as specified in [18, 19], allow to investigate the right tail of the loss distribution with any accuracy necessary for the case, i.e. to investigate the distribution tail so thoroughly, as required under the circumstances. Generally, it is reasonable to look for risk measures that are ideal for a particular private problem. Since all proposed risk measures have disadvantages and limitations in application, the choice of appropriate risk measure continues to be a highly discussed topic in risk management.

We offer upper limits of *VaR* and *ES* measures, and $VaR_p^{(t)}$ and $ES_p^{(t)}$. measures in this paper. In addition, using the results of Hürlimann [12], we get the values for the maximum of $VaR_p^{(t)}$ and $ES_p^{(t)}$ risk measures, when ignorance of the theory of loss allocation and the use of only the first few points of the theory of loss allocation. In addition, summarizing the consideration of Hürlimann [12], the author presented economic capital assessment in hedging losses above their lowest possible upper level using $ES_p^{(t)}$ risk measures.

UPPER LIMITS OF *VAR* AND *ES* RISK MEASURES WITH THE FIRST TWO MOMENTS IN LOSS DISTRIBUTION LAW

Let's start by defining the upper limit for normal *VaR* risk measure.

To this end, let's introduce inequality from paper [1] (see exercise 2.7.7).

Statement 1. (Basic inequality for *VaR* through first-order moment). Let $X > 0$ — is random value representing the amount of possible losses. Then the inequality

$VaR_p[X] \le \frac{E[X]}{1-p}$ is true for any p.

In the paper [18], group measures "*VaR* in the degree *t*" were introduced, where *t* — any real number $t \ge 1$, denoted as $VaR_p^{(t)}[X]$.

Any real number $t \ge 1$ can be uniquely represented as:

$t = k + \alpha$, where k — natural number, and α — real number, and $0 \le \alpha < 1$. Obviously, k is the integer part of *t*, and α — its fractional part.

In the paper [18], the following formula was proved, expressing them through the usual *VaR* risk measures.

For *VaR* risk measure in any real degree $t \ge 1$, $VaR_p^{(t)}[X]$ the following formula is valid:

$$VaR_p^{(t)}[X] = VaR_{1-(1-p)^k(1-\alpha p)}[X]. \quad (1)$$

Thus, in order to calculate $VaR_p^{(n)}$, risk measure, need to calculate the *VaR* risk measure with a confidence probability $1-(1-p)^k(1-\alpha p)$.

Then, given formula (1) and statement 1, the following statement is true.

Statement 2. (Basic inequality for $VaR_p^{(t)}$ through first-order moment). Let $X > 0$ — is random value representing the amount of possible losses. Then the inequality

$$VaR_p^{(t)}[X] \le \frac{E[X]}{(1-p)^k(1-\alpha p)}$$

is true for any p.

Let's define the upper limit for *ES* risk measure.

In paper [1] (see exercise 2.7.15) it is argued that for any random value of *X* loss with an average μ and variance σ^2 the following inequalit $ES_p[X] \le \mu + \sigma\sqrt{p(1-p)}$ is true for any p.

However, it is easy to understand that for a random quantity *X* with a random probability distribution such an inequality cannot be true, as $ES_p[X]$ risk measure when the confidence probability p is closer to 1, the value of the measure should approach indefinitely the upper limit of the loss allocation. In particular, for probability distributions of loss with infinite value (for example, for a normal distribution) when the confidence probability p is closer to 1, the value of the measure must be infinitely close to $+\infty$. However, in the given inequality, the upper limit for $ES_p[X]$ tends to the finite value μ, at the approach of the confidence probability p to 1, which, in the case of an arbitrary distribution, cannot be.

However, interesting is the fact of getting any right inequality $ES_p[X]$, that valid for any p.

Further, we prove the following statement.

Statement 3. (Basic inequality for *ES* through first and second order moments).

Let *X* — random value representing the value of possible losses with average μ and variance σ^2. Then inequality

$$ES_p[X] \le \mu + \frac{2\sigma}{\sqrt{1-p}}$$

is true for any p.

Verification. As is known (see [1]), $ES_p[X]$ risk measure is expressed through the appropriate *VaR* values as follows:

$$ES_p[X] = \frac{1}{1-p}\int_p^1 VaR_q[X]dq.$$

According to the *VaR* definition we have:

$\Pr[X \le VaR_q[X]] = q$, which is equivalent to

$$\Pr[\frac{X-\mu}{\sigma} \le \frac{VaR_q[X]-\mu}{\sigma}] = q.$$

Then, if we mark the value of the respective normalized random value by

$X^{(0.1)} = \frac{X-\mu}{\sigma}$, we get:

$$\Pr[X^{(0.1)} \le \frac{VaR_q[X]-\mu}{\sigma}] = q, \quad (2)$$

where $E[X^{(0.1)}] = 0$ и $\sigma[X^{(0.1)}] = 1$.

It follows from the equation (2), that $k_q^{(0.1)} = \frac{VaR_q[X]-\mu}{\sigma}$ value is a quantile of standardized random $X^{(0.1)}$ value with confidence probability *q*. The last ratio is followed by the following representation for *VaR*:

$$VaR_q[X] = \mu + k_q^{(0.1)}\sigma. \quad (3)$$

Using (3), we get the following equation for $ES_p[X]$:

$$ES_p[X] = \mu + \frac{\sigma}{1-p}\int_p^1 k_q^{(0.1)}dq. \quad (4)$$

Next, we got an estimate for quantil $k_q^{(0.1)}$. Take the second Chebyshev's inequality [20] which states that for any random quantity *X*, $\Pr[|X-\mu|>\varepsilon] \le \frac{\sigma^2[X]}{\varepsilon^2}$ is true for any positive ε.

Applying this inequality to the normalized random valu $X^{(0.1)}$ and choosing $\varepsilon = k_q^{(0.1)}$, we get: $\Pr[|X^{(0.1)}|>k_q^{(0.1)}] \le \frac{1}{(k_q^{(0.1)})^2}$, which implies that

$$1-q \le \Pr[X^{(0.1)} > k_q^{(0.1)}] + \Pr[X^{(0.1)} < -k_q^{(0.1)}] = \Pr[|X^{(0.1)}| > k_q^{(0.1)}] \le \frac{1}{(k_q^{(0.1)})^2}.$$

From the latter inequality follows the estimate for quantile:

$$|k_q^{(0.1)}| \le \frac{1}{\sqrt{1-q}}. \quad (5)$$

Using inequality (6), we get:

$$\int_p^1 k_q^{(0.1)}dq \le \int_p^1 \frac{dq}{\sqrt{1-q}} = -2\sqrt{1-q}\,\Big|_p^1 = 2\sqrt{1-p},$$

from where using (4) we get: $ES_p[X] \le \mu + \frac{2\sigma}{\sqrt{1-p}}$, which was to be proved.

Note that the right side of the inequality is close to $+\infty$, at the approach of the confidence probability *p* to 1 not contradicting that the $ES_p[X]$ risk measures for infinite value distributions approach the confidence probability *p* к 1 infinitely close to $+\infty$.

In the paper [19] group measures "*ES* in the degree *t*", where *t* — any real number $t \ge 1$, denoted as $ES_p^{(t)}[X]$.

Any real number $t \ge 1$ can be uniquely represented as: $t = k + \alpha$, where *k* — is a natural number, and α — real number, and $0 \le \alpha < 1$. Obviously, *k* is the integer part of *t*, and α is its fractional part.

Then for the risk measures from this group in [19] were found the next presentation, expressing them through the usual *ES* risk measures.

For *ES* risk measure in any real degree $t \ge 1$, $ES_p^{(t)}[X]$ the following formula is valid:

$$ES_p^{(t)}[X] = ES_{1-(1-p)^k(1-\alpha p)}[X]. \quad (6)$$

Thus, in order to calculate the $ES_p^{(t)}$, risk measure, need to calculate the *ES* risk measure with a confidence probability $1-(1-p)^k(1-\alpha p)$.

Statement 4. (Basic inequality for $ES_p^{(t)}$ through first and second order moments).

Let X — random value representing the amount of possible loss. Then the inequality

$$ES_p^{(t)}[X] \le \mu + \frac{2\sigma}{\sqrt{(1-p)^k(1-\alpha p)}}$$ is true for any p.

MAXIMUM *VAR* AND *ES* RISK MEASURES, $VaR^{(t)}$ AND $ES^{(t)}$ IN LIMITED DISTRIBUTIONS

In paper W. Hürlimann [12], the following result is given about maximum values of *VaR* and *ES* risk measures for random values, representing relevant risks with probability distributions with limited and fixed values of expected values and standard deviations.

We will introduce to consider a set of random variables.

Suppose that the value of the relevant distributions is the same as [*A, B*] segment, and by $D_2 = D_2([A,B];\mu,\sigma)$ denote the set of all random values *X* with [*A, B*] (*suppX* = [*A, B*]), with the expected value $E[X] = \mu$ and variance $D[X] = \sigma^2$.

In the paper [12] the following theorem has been proved (it is given using symbols in this paper).

Theorem 1 (Hürlimann W.). The maximum value of the *VaR* and risk measures on a set $D_2 = D_2([A,B];\mu,\sigma)$ is defined as follows:

Case 1: if $p \ge \frac{(B-\mu)^2}{\sigma^2+(B-\mu)^2}$, then $\max\limits_{X\in D_2}\{VaR_p[X]\} = \max\limits_{X\in D_2}\{ES_p[X]\} = B$.

Case 2: if $\frac{\sigma^2}{\sigma^2+(\mu-A)^2} \le p \le \frac{(B-\mu)^2}{\sigma^2+(B-\mu)^2}$, then $\max\limits_{X\in D_2}\{VaR_p[X]\} = \max\limits_{X\in D_2}\{ES_p[X]\} = \mu + \sqrt{\frac{p}{1-p}}\sigma$.

Case 3: if $p \le \frac{\sigma^2}{\sigma^2+(\mu-A)^2}$, then $\max\limits_{X\in D_2}\{VaR_p[X]\} = \mu + \frac{(\mu-A)(B-A)p-\sigma^2}{(B-A)(1-p)-(\mu-A)} \le$

$\le \max\limits_{X\in D_2}\{ES_p[X]\} = \mu + (\mu-A)(\frac{p}{1-p})$.

Comparing this statement with the measures upper *VaR* and *ES* risk measures given in statements 1 and 3, it is worth noting that the measures above the estimates of these risks in theorem 1, being maximal on a set of random variables (risks) $D_2 = D_2([A,B];\mu,\sigma)$, are more accurate, and estimates in statements 1 and 3 may be overstated in certain cases. However, the advantage of estimates in statements 1 and 3 is that they are valid for any random values (risks) with not necessarily limited value of the relevant probability distributions.

Let's move from the description of the maximum values of the respective $VaR^{(t)}$ and $ES^{(t)}$ risk measures and any real value $t \ge 1$ [18, 19].

Theorem 2. The maximum value of $VaR^{(t)}$ and $ES^{(t)}$ risk measures on a set of random values $D_2 = D_2([A,B];\mu,\sigma)$ is determined as follows: imagine the real number *t* as $t = m + \alpha$, where m — natural number, and α — real number within $0 < \alpha \le 1$.

Case 1: if $p \ge p_0$, where p_0 — unique solution to the equation $(1-p)^m(1-\alpha p) = \frac{\sigma^2}{\sigma^2+(B-\mu)^2}$, then

$\max\limits_{X\in D_2}\{VaR_p^{(t)}[X]\} = \max\limits_{X\in D_2}\{ES_p^{(t)}[X]\} = B$.

Case 2: if $p_1 \le p \le p_0$, where p_1 — unique solution to the equation

$$(1-p)^m(1-\alpha p) = \frac{(\mu - A)^2}{\sigma^2 + (\mu - A)^2}, \text{ then}$$

$$\max_{X \in D_2}\{VaR_p^{(t)}[X]\} = \max_{X \in D_2}\{ES_p^{(t)}[X]\} = \mu + \sqrt{\frac{1-(1-p)^m(1-\alpha p)}{(1-p)^m(1-\alpha p)}}\sigma.$$

Case 3: if $p \le p_1$, then $\max_{X \in D_2}\{VaR_p^{(t)}[X]\} = \mu + \frac{(\mu - A)(B-A)(1-(1-p)^m(1-\alpha p)) - \sigma^2}{(B-A)(1-p)^m(1-\alpha p) - (\mu - A)} \le$

$$\le \max_{X \in D_2}\{ES_p^{(t)}[X]\} = \mu + (\mu - A)\frac{1-(1-p)^m(1-\alpha p)}{(1-p)^m(1-\alpha p)}.$$

Verification. Taking into account the formulas connecting the $VaR^{(t)}$ and $ES^{(t)}$ risk measures and the usual *VaR* and *ES* risk measures, $VaR_p^{(t)}[X] = VaR_{1-(1-p)^m(1-\alpha p)}[X]$ and $ES_p^{(t)}[X] = ES_{1-(1-p)^m(1-\alpha p)}[X]$, we understand that to obtain theorem 2 statements it is enough in theorem 1 to replace p on $1-(1-p)^m(1-\alpha p)$.

Then Case 1 is realized with the confidence probability values satisfying the condition

$1-(1-p)^m(1-\alpha p) \ge \frac{(B-\mu)^2}{\sigma^2 + (B-\mu)^2}$, which is equivalent to the condition

$$(1-p)^m(1-\alpha p) \le \frac{\sigma^2}{\sigma^2 + (B-\mu)^2}, \tag{7}$$

where $\frac{\sigma^2}{\sigma^2 + (B-\mu)^2} \le 1$.

For the study of the set of solutions of the last inequality, we consider the function: $f(p) = (1-p)^m(1-\alpha p)$.

Then $f'(p) = -m(1-p)^{m-1}(1-\alpha p) - \alpha(1-p)^m = -(1-p)^{m-1}[m(1-\alpha p) + \alpha(1-p)) =$

$$= (1-p)^{m-1}[\alpha p(1+m) - m - \alpha] = (1-p)^{m-1}\alpha(m+1)[p - \frac{m+\alpha}{\alpha(m+1)}]. \tag{8}$$

However, it is easy to verify that inequality $p \le \frac{m+\alpha}{\alpha(m+1)}$, is always true, because inequality $\frac{m+\alpha}{\alpha(m+1)} \ge 1$, is true also, that is equivalent $m(1-\alpha) \ge 0$.

Then it follows from (8) that $f'(p) \le 0$, and therefore, the function $f(p) = (1-p)^m(1-\alpha p)$ is non-increasing, and $f(0) = 1 \ge \frac{\sigma^2}{\sigma^2 + (B-\mu)^2}$.

It follows that the equation $(1-p)^m(1-\alpha p) = \frac{\sigma^2}{\sigma^2 + (B-\mu)^2}$ has a single solution p_0 ($0 \le p_0 \le 1$), and $p \ge p_0$ inequality is executed (7).

This theorem is referenced in Case 1.

Case 2 is realized with confidence probability values satisfying the conditions $\frac{\sigma^2}{\sigma^2 + (\mu - A)^2} \le 1-(1-p)^m(1-\alpha p) \le \frac{(B-\mu)^2}{\sigma^2 + (B-\mu)^2}$, which are equivalent to the conditions

$$\frac{\sigma^2}{\sigma^2+(B-\mu)^2} \le (1-p)^m(1-\alpha p) \le \frac{(\mu-A)^2}{\sigma^2+(\mu-A)^2}. \quad (9)$$

Similarly, Case 1 proves the existence of a single value p_1 $(0 \le p_1 \le p_0 \le 1)$, solutions of the equation $(1-p)^m(1-\alpha p) = \frac{(\mu-A)^2}{\sigma^2+(\mu-A)^2}$, and $p_1 \le p \le p_0$ inequality is executed (9).

This theorem is referenced in Case 2.

And Case 3 is realized with the confidence probability values satisfying the condition $1-(1-p)^m(1-\alpha p) \le \frac{\sigma^2}{\sigma^2+(\mu-A)^2}$, which is equivalent to the condition

$$(1-p)^m(1-\alpha p) \ge \frac{(\mu-A)^2}{\sigma^2+(\mu-A)^2}. \quad (10)$$

From previous versions, it follows that an inequality $p \le p_1$ is executed (10).

This theorem is referenced in Case 3.

Comparing this statement with the estimates of upper $VaR^{(t)}$ and $ES^{(t)}$, risk measures, given in statements 2 and 4, it is worth noting that the estimates from upper limits of these risk measures in theorem 2 are maximal on a set of random variables (risks) $D_2 = D_2([A,B];\mu,\sigma)$, and are more accurate, but estimates in statements 2 and 4 might prove to be higher than required. However, the advantage of estimates in statements 3 and 4 is that they are true for any random values (risks) with not necessarily a limited value of the relevant probability distributions.

ESTIMATION OF ECONOMIC CAPITAL TO HEDGE LOSSES ABOVE THEIR LOWEST POSSIBLE UPPER LIMIT

Let's take some criteria for random quantities belonging to a set $D_2 = D_2([A,B];\mu,\sigma)$. Note that this criteria for $\mu = 0$ and $\sigma = 1$ is used in paper by W. Hürlimann [12].

Criteria. For any random quantity belonging to a set $D_2 = D_2([A,B];\mu,\sigma)$, the following relationships between parameters that describing a set are valid:

a) $A \le \mu \le B$;
b) $\sigma^2 \le (B-\mu)(\mu-A)$.

Verification. The first inequality follows from taking the expectation in the following random inequalities that are true with probability 1 for all $X \in D_2([A,B];\mu,\sigma)$: $A \le X \le B$.

To prove the inequality *b)* let's go to take the mathematical expectation in the following inequality, which is true with probability 1 for all $X \in D_2([A,B];\mu,\sigma)$: $(B-X)(X-A) \ge 0$.

Then we have:

$$B\mu - E(X^2) - AB + A\mu \ge 0, \text{ or}$$

$$E(X^2) - \mu^2 \le A\mu + \mu B - AB - \mu^2, \text{ i.e.}$$

$$\sigma^2 \le (B-\mu)(\mu-A).$$

Consider a company exposed to loss risks represented by random values belonging to a set $D_2 = D_2([0,B];\mu,\sigma)$ and try to estimate a minimum level of loss *B*.

From item *b)* it follows that $\sigma^2 \le \mu(B-\mu)$, which it follows that

$$B \ge \mu + \frac{\sigma^2}{\mu} = \mu(1+\frac{\sigma^2}{\mu^2}).$$

By introducing the value $k = \frac{\sigma}{\mu}$ — coefficient of variation, we get the following limit for the value *B* of the maximum possible loss: $B \ge \mu(1+k^2)$.

Thus, the maximum possible loss of the company cannot be less than the value $\mu(1+k^2)$.

In these circumstances, it is necessary to hedge the company from losses exceeding this amount, using derivatives or buying appropriate insurance.

Furthermore, we assume, for a start, that risk capital is calculated using a ES_p, risk measure and, for example, be assumed to be equal $\max_{X\in D_2}\{ES_p[X]\}$. (From the point of view of practice it is necessary to calculate it as a certain percentage of $\max_{X\in D_2}\{ES_p[X]\}$, but we focus on this assumption for simplicity).

Remember that in these assumptions from theorem 1 it follows that:

$$\max_{X\in D_2}\{ES_p[X]\} = \begin{cases} B, p \ge \frac{(B-\mu)^2}{\sigma^2+(B-\mu)^2}, \\ \mu + \sqrt{\frac{p}{1-p}}\sigma, \frac{k^2}{1+k^2} \le p < \frac{(B-\mu)^2}{\sigma^2+(B-\mu)^2}, \\ (1+\frac{p}{1-p})\mu, p < \frac{k^2}{1+k^2}, \end{cases}$$

when
$B = \mu(1+k^2)$ we have:

$$\frac{(B-\mu)^2}{\sigma^2+(B-\mu)^2} = \frac{(\mu(1+k^2)-\mu)^2}{\sigma^2+(\mu(1+k^2)-\mu)^2} = \frac{\mu^2k^4}{\sigma^2+\mu^2k^4} = \frac{\sigma^2k^2}{\sigma^2+\sigma^2k^2} = \frac{k^2}{1+k^2}.$$

So we have the following compact expression for venture capital:

$$\max_{X\in D_2}\{ES_p[X]\} = \begin{cases} (1+k^2)\mu, p \ge \frac{k^2}{1+k^2}, \\ (1+\frac{p}{1-p})\mu, p < \frac{k^2}{1+k^2}. \end{cases} \quad (11)$$

This scheme belongs to W. Hürlimann [12].

To understand the degree of caution when applying the described risk capital estimate, consider the numerical example.

Suppose that the confidence probability *p*, with which the ES_p, risk measure is estimated in this company equals *p* = 0.95. In addition, select the parameter value $\mu = 10\ \textit{ed.}$ and, by changing the parameter of model σ (and therefore *k*), we will find out which of the conditions for equality (11) will

Table 1

Calculation of Risk Capital $\max_{X \in D_2}\{ES_{0.95}[X]\}$ at Different Values of Parameters σ, k

No.	σ, k	$\frac{k^2}{1+k^2}$	Criterion	$\max_{X \in D_2}\{ES_{0.95}[X]\}$
1	$\sigma = 2un., \quad k = 0.2$	0.039	$p \geq \frac{k^2}{1+k^2}$	$B = 10.4$
2	$\sigma = 5un., \quad k = 0.5$	0.2	$p \geq \frac{k^2}{1+k^2}$	$B = 12.5$
3	$\sigma = 10un., \quad k = 1$	0.5	$p \geq \frac{k^2}{1+k^2}$	$B = 20$
4	$\sigma = 20un., \quad k = 2$	0.8	$p \geq \frac{k^2}{1+k^2}$	$B = 50$
5	$\sigma = 50un., \quad k = 5$	0.96	$p < \frac{k^2}{1+k^2}$	$(1+\frac{p}{1-p})\mu = 190 < B = 260$

Source: Designed and compiled by the author.

be fulfilled — and calculate the risk capital value accordingly. Results of the calculations are presented in *Table 1*.

We can see that with relatively small coefficients of variation k (the first four cases), which results in a relatively small non-hedged part of the possible losses $B = \mu(1+k^2)$, in this model the risk capital is value at the maximum equal to $B = \mu(1+k^2)$. However, in the case of large coefficients of variation (fifth case), the model determines the amount of risk capital required in the form of 190 un., the smaller part of possible losses that is not hedged, which is 260 un., i.e. significantly larger.

This involves how the coefficient of variation, which can range in value from some value between 2 and 5, is used to calculate how much economic capital should be included in the model. We determined this critical value of the coefficient of variation. Clearly, the change begins with implementation of the inequality $p < \frac{k^2}{1+k^2}$, which is equivalent to inequality $k > \sqrt{\frac{p}{1-p}} = \sqrt{\frac{0.95}{1-0.95}} \approx 4.36.$

Thus, at higher coefficients of variation, starting from a critical value of 4.36, the model determines the amount of risk capital required as the value of the smaller, non-hedged part of the potential losses.

We continue to consider our company, which was hedged from losses exceeding the value of $\mu(1+k^2)$, using derivatives or buying appropriate insurance.

In addition, suppose that the risk capital is calculated using the $ES_p^{(n)}$, risk measure, where n — natural number ($n > 1$), and, for example, be assumed to be equal $\max_{X \in D_2}\{ES_p^{(n)}[X]\}$.

We will remind that in these assumptions from theorem 2 it follows that:

$$\max_{X\in D_2}\{ES_p^{(n)}[X]\}=\begin{cases}B, p\geq 1-\sqrt[n]{\frac{\sigma^2}{\sigma^2+(B-\mu)^2}},\\ \mu+\sqrt{\frac{1-(1-p)^n}{(1-p)^n}}\sigma, 1-\sqrt[n]{\frac{1}{1+k^2}}\leq p<1-\sqrt[n]{\frac{\sigma^2}{\sigma^2+(B-\mu)^2}},\\ (1+\frac{1-(1-p)^n}{(1-p)^n})\mu, p<1-\sqrt[n]{\frac{1}{1+k^2}}.\end{cases}$$

However, in

$B=\mu(1+k^2)$ we get:

$$1-\sqrt[n]{\frac{\sigma^2}{\sigma^2+(B-\mu)^2}}=1-\sqrt[n]{\frac{\sigma^2}{\sigma^2+(\mu(1+k^2)-\mu)^2}}=1-\sqrt[n]{\frac{\sigma^2}{\sigma^2+\mu^2k^4}}=$$
$$=1-\sqrt[n]{\frac{\sigma^2}{\sigma^2+\sigma^2k^2}}=1-\sqrt[n]{\frac{1}{1+k^2}}.$$

We get the next compact equation for venture capital:

$$\max_{X\in D_2}\{ES_p^{(n)}[X]\}=\begin{cases}(1+k^2)\mu, p\geq 1-\sqrt[n]{\frac{1}{1+k^2}},\\ (1+\frac{(1-(1-p)^n}{(1-p)^n})\mu, p<1-\sqrt[n]{\frac{1}{1+k^2}}.\end{cases} \quad (12)$$

To understand the degree of caution with the application of the described assessment of risk capital, consider the numerical example.

Let's assume that for estimation of economic capital with model the $ES_p^{(n)}$ risk measure at $n = 2$, i.e. $ES_p^{(2)}$.

Suppose again that the confidence probability p, with which the $ES_p^{(2)}$, risk measure is assessed in this company equals $p = 0.95$. In addition, select the parameter $\mu=10$ *un.* and, by changing the parameter value of the model σ (and therefore k), we will find out which of the conditions in equality (12) will be fulfilled, and calculate the value of risk capital accordingly. Results of the calculations are presented in *Table 2*.

We see that for all the same values of the coefficients of variation k in this model with the application of $ES_{0.95}^{(2)}$ risk measure instead of $ES_{0.95}$ risk capital is valued at a maximum and equal $B=\mu(1+k^2)$. That is $ES_{0.95}^{(2)}$ risk measure is more cautious than $ES_{0.95}$.

Clearly, there will be a change in the way economic capital is measured in this model, depending on the value of the coefficient of variation, ranging with some value. We defined this critical value of the coefficient of variation. It is clear that change begins with the implementation of inequality

$p<1-\sqrt{\frac{1}{1+k^2}}$, which is equivalent to inequality

$$k>\sqrt{\frac{1-(1-p)^2}{(1-p)^2}}=\sqrt{\frac{1-(1-0.95)^2}{(1-0.95)^2}}\approx 19.98.$$

Table 2

Calculation of Risk Capital $\max_{X \in D_2}\{ES_{0,95}^{(2)}[X]\}$ at Different Values of Parameters σ, k

No.	σ, k	$1-\sqrt{\frac{1}{1+k^2}}$	Criterion	$\max_{X \in D_2}\{ES_{0,95}^{(2)}[X]\}$
1	$\sigma = 2un., \quad k = 0.2$	0.194	$p \geq 1-\sqrt{\frac{1}{1+k^2}}$	$B = 10.4$
2	$\sigma = 5un., \quad k = 0.5$	0.1055	$p \geq 1-\sqrt{\frac{1}{1+k^2}}$	$B = 12.5$
3	$\sigma = 10un., \quad k = 1$	0.29	$p \geq 1-\sqrt{\frac{1}{1+k^2}}$	$B = 20$
4	$\sigma = 20un., \quad k = 2$	0.553	$p \geq 1-\sqrt{\frac{1}{1+k^2}}$	$B = 50$
5	$\sigma = 50un., \quad k = 5$	0.804	$p \geq 1-\sqrt{\frac{1}{1+k^2}}$	$B = 260$

Source: Designed and compiled by the author.

At all such values *k*, the risk capital value is equal $\max_{X \in D_2}\{ES_{0.95}^{(2)}[X]\} = (1-\frac{1-(1-0.95)^2}{(1-0.95)^2})10 = 4000\ un.$, while the non-hedged part of possible losses even at $k = 20$ is equal $B = \mu(1+k^2) = 4010$.

Thus, at high coefficients of variation, starting from the critical value of 19.98, the model determines the amount of risk capital required in the form of the value of a smaller, non-hedged part of possible losses. That is, starting with such large coefficients of variation, and this model do not cautious as possible. Further, if the risk capital model based on $ES_{0.95}^{(3)}$, risk measure is applied, it turns out that the corresponding critical value of the coefficient of variation is even higher — about 89.44 etc.

Let's continue to consider our company, which was hedged against losses exceeding the $\mu(1+k^2)$, value, using derivatives or buying appropriate insurance.

In addition, suppose that risk capital is calculated using $ES_p^{(t)}$, risk measures, where *t* — real number ($t > 1$), and, for example, be assumed to be equal $\max_{X \in D_2}\{ES_p^{(t)}[X]\}$.

Consider the number *t* as $t = m + \alpha$, where *m* — natural number, and α real number when $0 < \alpha \leq 1$. Keep in mind that from these theorem 3 assumptions, it follows that

$$\max_{X\in D_2}\{ES_p^{(t)}[X]\}=\begin{cases}B, p\geq p_0,\\ \mu+\sqrt{\dfrac{1-(1-p)^m(1-\alpha p)}{(1-p)^m(1-\alpha p)}}\sigma, p_1\leq p<p_0,\\ (1+\dfrac{1-(1-p)^m(1-\alpha p)}{(1-p)^m(1-\alpha p)})\mu, p<p_1.\end{cases}$$

Remember, that p_0 — is unique solution to the equation

$$(1-p)^m(1-\alpha p)=\frac{\sigma^2}{\sigma^2+(B-\mu)^2},$$

and p_1 — is unique solution to the equation

$$(1-p)^m(1-\alpha p)=\frac{\mu^2}{\sigma^2+\mu^2}.$$

However, in $B=\mu(1+k^2)$ we get:

$$\frac{\sigma^2}{\sigma^2+(B-\mu)^2}=\frac{\sigma^2}{\sigma^2+(\mu(1+k^2)-\mu)^2}=\frac{\sigma^2}{\sigma^2+\mu^2k^4}=\frac{\sigma^2}{\sigma^2+\sigma^2k^2}=\frac{1}{1+k^2}=\frac{\mu^2}{\sigma^2+\mu^2},$$

i.e. $p_0=p_1$.

Therefore we have the following compact equation for venture capital:

$$\max_{X\in D_2}\{ES_p^{(t)}[X]\}=\begin{cases}(1+k^2)\mu, p\geq p_0,\\ (1+\dfrac{(1-(1-p)^m(1-\alpha p)}{(1-p)^m(1-\alpha p)})\mu, p<p_0.\end{cases} \quad (13)$$

To understand the degree of caution with the application of the described assessment of risk capital, consider the numerical example.

Suppose that the $ES_p^{(t)}$ risk measure is chosen to estimate the economic capital using this model with t = 1.5, i.e. $ES_p^{(1.5)}$.

Suppose again that the confidence probability p, with which the $ES_p^{(1.5)}$, risk measure is assessed in this company equals p = 0.95. In addition, select the value of the parameter $\mu=10$ *un.* and, by changing the parameter of the model σ (and therefore k), we will find out which of the conditions in equality (13) will be fulfilled and calculate the value of risk capital accordingly.

Note that the choice in the formula (13) of an expression to calculate risk capital depends on whether the $(1-p)^m(1-\alpha p)$ value is greater or less than the $\frac{1}{1+k^2}$. However, in the present case m = 1 and $\alpha=0.5$, then $(1-p)^m(1-\alpha p)=0.02625$.

Results of the calculations are presented in *Table 3*.

We see that for all the same values of the coefficients of variation k in this model using the $ES_{0.95}^{(1.5)}$ risk measure, the risk capital is valued at the maximum equal to $B=\mu(1+k^2)$. That is, the $ES_{0.95}^{(1.5)}$ risk measure as well as $ES_{0.95}^{(2)}$, is more cautious than $ES_{0.95}$.

Table 3

Calculation of Risk Capital $\max\limits_{X \in D_2}\{ES_{0.95}^{(1.5)}[X]\}$ at Different Values of Parameters σ, k

No.	σ, k	$\frac{1}{1+k^2}$	Criterion	$\max\limits_{X \in D_2}\{ES_{0.95}^{(1.5)}[X]\}$
1	$\sigma = 2un., \quad k = 0.2$	0.96	$0.02625 < \frac{1}{1+k^2}$	$B = 12.5$
2	$\sigma = 5un., \quad k = 0.5$	0.8	$0.02625 < \frac{1}{1+k^2}$	$B = 12.5$
3	$\sigma = 10un., \quad k = 1$	0.5	$0.02625 < \frac{1}{1+k^2}$	$B = 20$
4	$\sigma = 20un., \quad k = 2$	0.2	$0.02625 < \frac{1}{1+k^2}$	$B = 50$
5	$\sigma = 50un., \quad k = 5$	0.039	$0.02625 < \frac{1}{1+k^2}$	$B = 260$

Source: Designed and compiled by the author.

Table 4

Calculation of Risk Capital $\max\limits_{X \in D_2}\{ES_{0.95}^{(1.2)}[X]\}$ at Different Values of Parameters σ, k

No.	σ, k	$\frac{1}{1+k^2}$	Criterion	$\max\limits_{X \in D_2}\{ES_{0.95}^{(1.2)}[X]\}$
1	$\sigma = 2un., \quad k = 0.2$	0.96	$0.0405 < \frac{1}{1+k^2}$	$B = 12.5$
2	$\sigma = 5un., \quad k = 0.5$	0.8	$0.0405 < \frac{1}{1+k^2}$	$B = 12.5$
3	$\sigma = 10un., \quad k = 1$	0.5	$0.0405 < \frac{1}{1+k^2}$	$B = 20$
4	$\sigma = 20un., \quad k = 2$	0.2	$0.0405 < \frac{1}{1+k^2}$	$B = 50$
5	$\sigma = 50un., \quad k = 5$	0.039	$0.0405 < \frac{1}{1+k^2}$	$(1+\frac{1-(1-0.95)(1-0.2 \cdot 0.95)}{(1-0.95)(1-0,2 \cdot 0.95)}) \cdot 10 = = 246,9$

Source: Designed and compiled by the author.

Now suppose that the $ES_p^{(t)}$ risk measure is chosen to estimate economic capital using this model, when $t = 1.2$, i.e. $ES_p^{(1.2)}$.

Suppose again that the confidence probability p, with which $ES_p^{(1.2)}$, risk measure is assessed in this company equals $p = 0.95$. In addition, we choose the parameter $\mu = 10\ un.$, and by changing the parameter value of the model σ (and therefore *k*), we will find out which of the conditions in equality (13) will be fulfilled and calculate the risk capital value accordingly.

Note that the choice in the formula (13) of an expression to calculate risk capital depends on whether the $(1-p)^m(1-\alpha p)$ value is larger or smaller than the $\frac{1}{1+k^2}$ value. However, in the present case $m = 1$ and $\alpha = 0.2$, and therefore $(1-p)^m(1-\alpha p) = 0.0405$.

Results of the calculations are presented in *Table 4*.

We observe that with relatively small coefficients of variation *k* (the first four cases), which results in a relatively small non-hedged part of the possible losses $B = \mu(1+k^2)$, in this model the risk capital is valued at the maximum equal to $B = \mu(1+k^2)$. However, in the case of large coefficients of variation (fifth case), the model determines the amount of risk capital required in the form of 246.9 un., smaller than the non-hedged part of the possible losses, which is 260 un., i.e. significantly larger.

It is clear that this change in the behavior of the measurement of economic capital in a given model, depending on the value of the coefficient of variation, occurs at some value between 2 and 5. And risk capital valuation models using $ES_p^{(t)}$ risk measures at $t \geq 1.5$ are much more cautious than the corresponding models a $t \leq 1.2$ and at the model parameter *t*, there is also some critical value $0.2 < t_0 < 1.5$, where there is a transition from one policy (less cautious) selection of risk capital to another (more cautious).

MAXIMUM *VAR* AND *ES* RISK MEASURES, $VaR^{(t)}$ AND $ES^{(t)}$ IN UNLIMITED DISTRIBUTIONS

In the paper of W. Hürlimann [12], the following result is given about the maximum values of *VaR* and *ES* risk measures for random values, representing relevant risks with probability distributions with *unlimited* and fixed values of expected values and standard deviations, coefficients of asymmetry and excesses.

That is, it focuses on the set $D_4((-\infty,\infty);\mu,\sigma,\gamma,\gamma_2)$ random quantities with values on $(-\infty,\infty)$ with average μ, variance σ^2, asymmetry γ and excesses γ_2. In all phases, the following extra variables will be used:

$$\Delta = 2+\gamma_2-\gamma^2,\ \ c = \frac{1}{2}(\gamma-\sqrt{4+\gamma^2}),\ \bar{c} = -c^{-1} = \frac{1}{2}(\gamma+\sqrt{4+\gamma^2}). \qquad (14)$$

The following theorem is proved in paper [12] (is used in the article with symbols).

Theorem 3. The maximum value of *VaR* for set D_4 is equal

$$\max_{X\in D_4}\{VaR_p[X]\} = \mu + x_p\sigma,$$

where x_p — quantile standardised maximum distribution $F^{(4)}_{ST,\max}(x)$ The following cases provide it:

Case 1: $p \geq 1-P(\bar{c}) = \frac{1}{2}(1+\frac{\gamma}{\sqrt{4+\gamma^2}}),\ p(x_p) = 1-p.$

Case 2: $p < 1-P(\bar{c}) = \frac{1}{2}(1+\frac{\gamma}{\sqrt{4+\gamma^2}}),\ p(\psi(x_p)) = p,$

where $\psi(x)$ and $p(x)$ functions defined as:

$$\psi(u) = \frac{1}{2}(\frac{A(u)-\sqrt{A(u)^2+4q(u)B(u)}}{q(u)}), \qquad (15)$$

$$A(u)=\gamma q(u)+\Delta u,\ B(u)=\Delta+q(u),\ q(u)=1+\gamma u-u^2, \tag{16}$$

$$P(u)=\frac{\Delta}{q(u)^2+\Delta(1+u^2)}. \tag{17}$$

Comparing this statement with the upper limits of *VaR* risk measure given in statement 1, it is worth noting that the upper evaluation of this risk measure in the theorem, being maximum on a set of random variables (risks) $D_4((-\infty,\infty);\mu,\sigma,\gamma,\gamma_2)$, is more accurate, and the rating in statement 1 may be overstated in certain cases. However, the advantage of an estimate in statement 1 is that it is true for any random values (risks) with a fixed expected value but with arbitrary values of standard deviation, asymmetry, and excesses, whereas the assessment in theorem 4 is valid with fixed values of standard deviation, coefficients of asymmetry, and excesses. In addition, the maximal upper estimator algorithm in theorem 4 requires numerical methods because there is no straight formula to calculate it, whereas the estimate according to proposition is extremely simple.

Proceed to a description of the maximum values of the respective $VaR^{(t)}$ risk measures at any valid value $t\geq 1$ (see [18, 19]).

Theorem 4. The maximum value $VaR^{(t)}$ on a set of random variables D_4 is defined as follows: let's suppose the real number *t* as $t=m+\alpha$, where *m* — natural number, and α — real number within $0<\alpha\leq 1$.

Then $\max\limits_{X\in D_4}\{VaR_p^{(t)}[X]\}=\mu+x_{1-(1-p)^m(1-\alpha p)}\sigma$, where x_p — quantile standardized maximum distribution $F_{ST,\max}^{(4)}(x)$ it is obtained as follows:

even if p_0 is unique solution to the equation $(1-p)^m(1-\alpha p)=\frac{1}{2}(1-\frac{\gamma}{\sqrt{4+\gamma^2}})$, then:

Case 1: if $p\geq p_0$, then $P(x_{1-(1-p)^m(1-\alpha p)})=(1-p)^m(1-\alpha p)$,

Case 2: if $p<p_0$, then $P(\psi(x_{1-(1-p)^m(1-\alpha p)}))=1-(1-p)^m(1-\alpha p)$.

Verification. Given the formula linking $VaR^{(t)}$ risk measures to the usual *VaR* risk measure: $VaR_p^{(t)}[X]=VaR_{1-(1-p)^m(1-\alpha p)}[X]$, we understand that to obtain theorem 2, it is sufficient in theorem 1 to replace *p* by $1-(1-p)^m(1-\alpha p)$.

Then case 1 is realized with the confidence probability values satisfying the condition

$1-(1-p)^m(1-\alpha p)\geq\frac{1}{2}(1+\frac{\gamma}{\sqrt{4+\gamma^2}})$, which is equivalent to the condition

$$(1-p)^m(1-\alpha p)\leq\frac{1}{2}(1-\frac{\gamma}{\sqrt{4+\gamma^2}}), \tag{18}$$

where $\frac{1}{2}(1-\frac{\gamma}{\sqrt{4+\gamma^2}})\leq 1$.

Theorem 3's proof, then, also establishes the existence and originality of the solution p_0 equations $(1-p)^m(1-\alpha p)=\frac{1}{2}(1-\frac{\gamma}{\sqrt{4+\gamma^2}})$, with $p\geq p_0$ is inequality (18), and $p<p_0$ is inequality opposite to (18). The proof of theorem 4 follows consequently and from theorem 3.

The following theorem is proved in paper [12] (is used in the article with symbols).

Theorem 5. The maximum value of *ES* on the set D_4 equal

$$\max_{X \in D_4}\{ES_p[X]\} = \mu + \{d(y_p) + \frac{1}{1-p}(\pi^{(4)}_{\max} \circ d)(y_p)\}\sigma,$$

where quantile of the maximum distribution $F^{(4)}_{SL,\max}(x)$ of the standardised stop-loss of order (see [29]) is derived from the following equations:

Case 1: if $p \geq 1 - P(\bar{c})$, then $P(y_p) = 1 - p$,

Case 2: if $p < 1 - P(\bar{c})$, then $P(y_p) = p$,

where $P(x)$ is determined from (18),

$$d(x) = \frac{1}{2}\frac{\{\phi(x,\psi(x)) - x\}\{x + \psi(x)\} + 2x\{\psi(x) - x\}}{\{\phi(x,\psi(x)) - x\} + \{\psi(x) - x\}},$$

$$\phi(u,v) = \frac{\gamma - u - v}{1 + uv},$$

$$\pi^{(4)}_{\max}(d(x)) = \begin{cases} P(x)(d(x) - x) - d(x), x < \bar{c} \\ P(x)(x - d(x)), x \geq \bar{c} \end{cases}$$

Proceed to a description of the maximum values of the respective $ES^{(t)}$ risk measures at any real value $t \geq 1$ (see [18, 19]).

Theorem 6. The maximum value $ES^{(t)}$ on a set of random variables D_4 is defined as follows: imagine the real number *t* as $t = m + \alpha$, where *m* — natural number, and α — real number within $0 < \alpha \leq 1$.

$$\max_{X \in D_4}\{ES^{(t)}_p[X]\} = \mu + \{d(y_{1-(1-p)^m(1-\alpha p)}) + \frac{1}{(1-p)^m(1-\alpha p)}(\pi^{(4)}_{\max} \circ d)(y_{1-(1-p)^m(1-\alpha p)})\}\sigma,$$

where $d(y_p)$ — where quantile of the maximum distribution $F^{(4)}_{SL,\max}(x)$ of the standardised stop-loss of order (see [12]) is derived from the following equations.

Even if p_0 — is unique solution to the equation $(1-p)^m(1-\alpha p) = P(\bar{c})$.

Then:

Case 1: if $p \geq p_0$, then $P(y_{1-(1-p)^m(1-\alpha p)}) = (1-p)^m(1-\alpha p)$,

Case 2: if $p < p_0$, then $P(y_{1-(1-p)^m(1-\alpha p)}) = 1 - (1-p)^m(1-\alpha p)$.

Verification. Given the formula linking $ES^{(t)}$ risk measures to the usual *ES* risk measure, $ES^{(t)}_p[X] = ES_{1-(1-p)^m(1-\alpha p)}[X]$, we understand that to obtain theorem 2, it is sufficient in theorem 1 to replace *p* by $1-(1-p)^m(1-\alpha p)$.

Then case 1 is realized with the confidence probability values satisfying the condition $1-(1-p)^m(1-\alpha p) \geq 1 - P(\bar{c})$, which is equivalent to the condition $(1-p)^m(1-\alpha p) \leq P(\bar{c})$. (19)

Theorem 3's proof, then, also establishes the existence and originality of the solution p_0 of equation $(1-p)^m(1-\alpha p) = P(\bar{c})$, with $p \geq p_0$ is inequality (19), and $p < p_0$ is inequality opposite to (19). The proof of theorem 7 follows consequently and from theorem 8.

CONCLUSION

Exploring the upper limits of various risk measures, including catastrophic risk measures, is of both scientific and practical interest. They are practical for quick risk assessments, which are easy to apply if the upper limits have clear and straightforward expressions. Cases when they are articulated just after the first few moments of the rule of loss allocation and do not require understanding of the law of distribution itself are particularly significant.

The paper examines the upper limits for known risk measures such as *VaR* risk measure, and the expected shortfall or imputed value at *ES* risk measure. Then the upper limits for measures introduced by the author in the scientific state of *VaR* catastrophic risks in the degree *t*, $VaR^{(t)}$ and *ES* in the degree *t*, $ES^{(t)}$.

The paper also describes the results of W. Hürlimann on the maximum values of *VaR* and *ES* risk measures, and with the application of these results the representations for maximum values of $VaR^{(t)}$ and $ES^{(t)}$ risk measures.

Using W. Hürlimann, in the paper provides an estimate of the value of economic capital by means of $ES^{(t)}$ risk measures depending on the coefficient of variation of losses in hedging losses above their lowest possible upper limits.

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ABOUT THE AUTHOR

Vigen B. Minasyan — Cand. Sci. (Phis.-Math.), Assoc. Prof., Head of Limitovskii corporate finance, investment design and evaluation department, Higher School of Finance and Management, Russian Presidential Academy of National Economy and Public Administration, Moscow, Russia
https://orcid.org/0000-0001-6393-145X
minasyanvb@ranepa.ru, minasyanvb@yandex.ru

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