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Transformation of Various Measures of Financial Risks with their Limitation on Outcomes Associated with Losses

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ABSTRACT
In assessing the risk of investing in various financial assets, risk management focuses on the analysis of the worst possible losses (the right tail of the loss distribution). At the same time, most often, when speaking about losses, it is assumed that losses can, in principle, take on negative values (which corresponds to receiving positive profits). However, there are many theoretical studies suggesting that losses take only positive values. Many risk managers use only a portion of the sample of data that corresponds to positive losses when assessing the relevant risk measures using the statistical method or the Monte Carlo method. The **purpose** of this paper is to study the transformation of risk estimates of various levels of catastrophicity with such a change in the space of elementary events, and hence the law of loss distribution. The paper uses methods of analysis of financial risks of various levels of catastrophicity, including methods developed in the author's previous papers. As a **result** of the study, it turned out that with such a transformation of the random value of losses, all the most important estimates are significantly transformed with the help of risk measures of various catastrophicity. The author **concludes** that the theoretical conclusions of the work will also contribute to a more conscious understanding of the theoretical results and the results of practical risk assessments, depending on the basis on which this assessment was made: allowing losses to accept negative values or focusing only on their positive values.
Keywords: transformation of risk measures; limitation of risk measures on losses; catastrophic financial risks; risk measures distortion of expectation; coherent measures of financial risks; measures of risk *VaR* in degree *t*; risk measures *ES* to the power *t*

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INTRODUCTION

The main subject of research in both risk management and insurance business are risks that are modeled in the form of random X values of interest to risk managers or actuaries. Assessment and management of these risks are the main tasks that are addressed by these specialists. From the point of view of risk analysis, the value of the adverse values of X, and the extent of their possibilities, which are determined by the law of the distribution of the value X are of interest. Often by the random value that acts on the relevant risk, we understand the value of the profit— and then the unfavorable values are represented by the negative values of the gain; or the amount of the losses — and so the undesirable value is presented as the positive value of losses. But this is most often assumed that the value of X can, in principle, take any both positive and negative values. In this paper we will consider that the value X represents the losses, and its positive values will be interpreted as losses and the negative values — as the corresponding profits. After all, initially, engaging in business, the entrepreneur hopes to get a certain profit, but understands that everything can end up fixing losses, which is perceived as a corresponding manifestation of risks. Therefore, we assume that the value of losses X is a random value distributed across the line $(-\infty,+\infty)$.

Sometimes researchers study the losses, assuming a priori they are non-negative, $X \geq 0$ (for example, [1]). In fact, investments can result in both profits and losses. And then the non-negative assumption of losses means that the researcher or company risk manager in the practice of risk analysis focuses only on the results that lead to losses. In this study, we would like to emphasize the fact that the probability distribution for this value of losses will differ from the probability distribution for the value of losses distributed across the line $(-\infty,+\infty)$. And accordingly, risk assessments with one and the other representation of the random value representing the risk will differ sometimes significantly. The study of the extent to which the risk assessment may change from a single representation of a corresponding random value as distributed throughout the line $(-\infty,+\infty)$ to a representation as a random value assuming that only its positive values are realized has not been adequately addressed. We tried to examine this problem in detail.

RELATION OF THE LAW OF THE DISTRIBUTION OF THE RANDOM VALUE *X* WITH IT'S LIMITATION TO POSITIVE VALUES ONLY

Let us be interested in the risk of losses associated with the values of the random value *X*, taking values on the entire line $(-\infty,+\infty)$. Select the distribution function through $F_X(x)$, $x\in(-\infty,+\infty)$. As is known, $F_X(x)=P[X<x]$ (via $P[A]$ through indicated the probability of a random event *A*). Suppose there is a continuous density of the distribution of the value *X*, $f_X(x)$, $x\in(-\infty,+\infty)$. At the same time $f_X(x)=F_X'(x)$.

Suppose the researcher focuses only on the outcomes associated with real losses when $X \geqslant 0$. At the same time, this concentration leads to two different approaches to the study of the losses.

1. At the time interest is reduced to the replacement in the study of *X* values by the values X_+, associated with the random value of *X* as follows:

$$X_+=\begin{cases}X, if\ X\ge 0\\ 0, if\ X<0.\end{cases}$$

Note that the value X_+ can also be represented as follows: $X_+=XI_{\{X\ge 0\}}$, where $I_{\{A\}}$ is indicated by the indicator function of the event *A*:

$$I_{\{A\}}=\begin{cases}1, if\ event\ A\ is\ executed\\ 0, if\ event\ A\ is\ not\ executed\end{cases}$$

It is intuitively apparent that the characteristics of the X_+ value distribution will differ from the corresponding characteristic of the random *X* distribution. Let's try to get expressions for the distribution function and density X_+ through the corresponding characteristics for *X*.

If $F_{X_+}(x)$ identify by the distribution function for X_+, i.e. $F_{X_+}(x)=P[X_+<x]$. Suppose that $x>0$, then by the formula of full probability we get:

$$F_{X_+}(x)=P[X_+<x\,|\,X\ge 0]P[X\ge 0]+\ P[X_+<x\,|\,X<0]P[X<0]=$$

$$=P[X_+<x\,|\,X\ge 0]P[X\ge 0]+\ P[0<x\,|\,X<0]P[X<0]$$

(here and then through $P[A\,|\,B]$ is indicated the conditional probability of event *A* on condition of event *B*), but, given that, obviously, $\Pr[0<x\,|\,X<0]=1$, we get:

$$F_{X_+}(x)=\frac{P[0\le X<x]}{P[X\ge 0]}P[X\ge 0]+\ P[X<0]=$$

$$=P[0\le X<x]+\ P[X<0]=\ P[X<x].$$

Then we get that at $x>0$, $F_{X_+}(x)=F_X(x)$, and it's obvious that when $x\le 0$, $F_{X_+}(x)=0$,

we get $$F_{X_+}(x)=\begin{cases}F_X(x), if\ x>0\\ 0, \quad if\ x\le 0.\end{cases} \tag{1}$$

Accordingly, if we mark through $f_{X_+}(x)$ the density of the distribution of the value X_+, because at $x>0$ $f_{X_+}(x)=F'_{X_+}(x)=f_X(x)$, we get the formula $f_{X_+}(x)=f_X(x)I_{\{x>0\}}+F_X(0)\delta(x)$, (2)
which is correct for all values of *x*, where $\delta(x)$ — is the known Dirac δ-function.

We see how the distribution function evolved when we switched from losses modeling as a random value *X* to random value X_+ modeling.

The resulting expressions are the following expressions for calculating the expected value and the dispersion of a random value X_+:

$$E[X_+]=\int_0^{+\infty} xf_X(x)dx,$$

$$D[X_+]=\int_0^{+\infty} x^2 f_X(x)dx-(\int_0^{+\infty} xf_X(x)dx)^2.$$

We see how the expressions of both the expected value and the dispersion were transformed as we moved from the modeling of the value of losses as a random value of *X* to its modeling in the form of the random value X_+. We understand that when assessing the risk in the form of a loss dispersion, the risk assessment when changing the representation of losses as *X* value to their representation as a value X_+ can vary considerably.

2. In other cases, interest is reduced to the replacement in the study of the distribution function of a random value *X* $F_X(x)$ with its conditional distribution with the condition of realization of the event of real loss, i.e. $X \geqslant 0$ (on the concept of the conditional-distribution function, see, for example, [1]), which we will indicate $F_X^+(x)$, where $F_X^+(x)=\Pr[X<x\,|\,X\ge 0]$.

It is intuitively obvious that this function of the conditional distribution of a value X will be different from the random distribution function of *X*, *X*, $F_X(x)$. Let's try to get expressions for $F_X^+(x)$ functions through $F_X(x)$.

Let's $x>0$, then it's obvious that $F_X^+(x)=\dfrac{P[0\le X<x]}{P[X\ge 0]}=\dfrac{P[0\le X<x]}{1-P[X<0]}$.

Using the definition of the distribution function, we get: $F_X^+(x)=\dfrac{F_X(x)-F_X(0)}{1-F_X(0)}$ at $x>0$.

Given that with all $x\le 0$, $F_X^+(x)=0$, we get the final expression of the function of the conditional distribution

$F_X^+(x)$ through the random value distribution function *X*:

$$F_X^+(x)=\begin{cases}\dfrac{F_X(x)-F_X(0)}{1-F_X(0)}, if\ x>0\\ 0, if\ x\le 0.\end{cases} \tag{3}$$

We see how the distribution function evolved in the transition from the function of unconditional distribution of the losses $F_X(x)$ to its modeling in the form of the given function of the conditions of distribution.

Accordingly, if we indicate through $f_X^+(x)$ the density of the conditional distribution of the value X with the condition X ⩾ 0, then $f_X^+(x) = (F_X^+)'(x)$, we get: $f_X^+(x) = \begin{cases} \dfrac{f_X(x)}{1-F_X(0)}, if\ x > 0 \\ 0, if\ x \le 0. \end{cases}$ (4)

We see how the density of the distribution evolved when we switched from the unconditional distribution of the losses $F_X(x)$ to its modeling in the form of a given function of the conditions of distribution.

The resulting expressions are obviously followed by the expressions for calculating the corresponding conditional expected value and the conditional dispersion of a random value X:

$$E^+[X] = \frac{1}{1-F(0)} \int_0^{+\infty} x f_X(x)dx,$$

$$D^+[X] = \frac{1}{1-F(0)} \int_0^{+\infty} x^2 f_X(x)dx - \frac{1}{(1-F(0))^2} (\int_0^{+\infty} x f_X(x)dx)^2.$$

We see how the expressions of both the expected value and the dispersion changed when moving from the loss model in both cases. In general, we understand that when assessing the risk in the form of a loss dispersion, the risk assessment when changing the representation of losses as a value X to their representation in both types may vary considerably.

RISK MEASURES VaR AND $VaR^{(t)}$, THEIR TRANSFORMATION OF VALUATIONS IN THE TRANSITION FROM REPRESENTATION OF LOSSES X TO REPRESENTATION IN MODIFIED VARIANTS

Value of risk (*VaR*) is one of the most commonly used risk measures in risk management and actuarial science (see, for example, [3–5]).

1. Consider how this risk measure is transformed when we move from the random loss value of X to the random value X_+, suggesting that the loss of X is non-negative ($X \ge 0$). Let's try to understand the relationship between the estimates.

Note that according to the determination of the risk measure *VaR* for value X with probability p, $VaR[X, p]$, the ratio is correct:

$$VaR[X, p] = F_X^{-1}(p),\ p \in (0,1].$$

Accordingly, for the risk measurement *VaR* in the risk assessment modeled by a random value X_+, $VaR[X_+, p]$, the ratio is correct:

$$VaR[X_+, p] = F_{X_+}^{-1}(p),\ p \in (0,1].$$

However, according to the formula (1) at $x > 0$ $F_{X_+}(x) = F_X(x)$ to determine $VaR_p(X_+)$ it is necessary for anyone $p \in (0,1]$ to solve the equation $F_X(x) = p$, and therefore a $p \in (0,1]$, which may be of interest, we get $VaR[X_+, p] = VaR[X, p]$.

Thus, when estimating the risk measure *VaR* for the value of losses for any positive trust probability, there is no change in the risk assessment when moving from the random value X to the value X_+.

In the papers [6] and [7] studied the disaster risk measures "*VaR* in degree *t*", $VaR^{(t)}[X, p]$, which proved to be a subset of well-examined expectation distortion risk measurements considered in large numbers of papers (see [8–13]).

For a risk measures *VaR* is in "degree" $t \geq 1$ $VaR^{(t)}[X, p]$, as demonstrated in paper [6], where the value of "degree" $t \geq 1$ is an random substantive number, which is presented as $t = m + \alpha$, where m — is its whole part, and α — is its fraction ($0 \leq \alpha < 1$), the formula $VaR^{(t)}[X, p] = VaR[X, 1-(1-p)^m(1-\alpha p)]$ is correct.

Of course, for the family of risk measures *VaR* in the "degree" $t \geq 1$ the formula $VaR^{(t)}[X_+, p] = VaR^{(t)}[X, p]$is correct.

The transition of losses modeling to random value X_+ instead of *X* — is quite harmless if you evaluate risks using $VaR^{(t)}[X, p]$ risk measure.

It is therefore clear that all other quantum risk measures, including *ES* and $ES^{(t)}$ (see [7]), will not be affected by this transformation.

Therefore, we will continue to focus only on the case of the second option of consideration of risk conversion with a focus on events of real losses, i.e. $X \geq 0$.

2. In this case, the study replaces the distribution function of the random value *X*, $F_X(x)$ with its conditional distribution, subject to the realization of the event of the real loss, i.e. $X \geq 0$, $F_X^+(x)$.

According to formula (3) at $x > 0$ $F_X^+(x) = \dfrac{F_X(x) - F_X(0)}{1 - F_X(0)}$.

It is essential to introduce the following definition of the measure of assumptive value at risk, which we shall designate $VaR^+[X, p]$ as the solution of the following equation: $F_X^+(x) = p$.

So for determination $VaR^+[X, p]$ it is necessary for anyone $p \in (0,1]$ to solve an equation $\dfrac{F_X(x) - F_X(0)}{1 - F_X(0)} = p$,

that is equivalent to the equation $F_X(x) = (1 - F_X(0))p + F_X(0)$, whose solution is presented as:

$x = (F_X^+)^{-1} = F_X^{-1}((1 - F_X(0))p + F_X(0))$, from which we get the following important equation:

$$VaR^+[X, p] = VaR[X, (1 - F_X(0))p + F_X(0)]. \quad (5)$$

Thus, in order to assess the risk measure conditional *VaR* for the value of losses, provided that they are realized ($X \geqslant 0$) with confidence probability *p*, it is sufficient to estimate the risk of *VaR* for the amount of the losses represented by the random value *X* with confidence probability $(1 - F_X(0))p + F_X(0)$.

Note. From the formula (5) it follows that for any $p \in (0,1]$ inequality is correct:

$VaR^+[X, p] \geq VaR[X, p]$, which follows that inequality $(1 - F_X(0))p + F_X(0) \geq p$ is correct.

Let us now examine how risk estimates represented by random values measured by catastrophic risk measures $VaR^{(t)}[X, p]$, are transformed by the transition to the corresponding risk conditional measures $VaR^{(t)}[X, p]$ at a condition of $X \geqslant 0$, which we will mark $VaR^{(t)+}[X, p]$.

The "degree" value of t in $VaR^{(t)}[X, p]$ is a natural number: $t = n$.

Let us note that in this case, as demonstrated in the paper [6], the correct formula

$$VaR^{(n)}[X, p] = VaR[X, 1 - (1 - p)^n], \quad (6)$$

which calculates the risk measurement *VaR* in degree *n* with probability *p*, to the calculation of the normal risk measured *VaR* with modified probability $1 - (1 - p)^n$.

Then, using the formulas (6) and (5), we get:

$$VaR^{(n)+}[X, p] = VaR^+[X, 1 - (1 - p)^n] =$$

$$= VaR[X, (1 - F_X(0))(1 - (1 - p)^n) + F_X(0)],$$

or $$VaR^{(n)+}[X,p] =$$

$$= VaR[X,(1-F_X(0))(1-(1-p)^n)+F_X(0)]. \quad (7)$$

Thus, in order to assess the *VaR* risk measurement of degree *n*, for the losses represented by the random value *X* with the condition $X \geqslant 0$ with confidence probability *p*, it is sufficient to estimate the risks measure *VaR* for the loss of the random value *X* with confidence probability $(1-F_X(0))(1-(1-p)^n)+F_X(0)$.

Note. It follows from the formulas (6) and (7) that for any $p \in (0,1]$ the correct inequality:

$VaR^{(n)+}[X,p] \geq VaR^{(n)}[X,p]$, which follows correct inequality $(1-F_X(0))(1-(1-p)^n)+F_X(0) \geq (1-(1-p)^n$.

There is considerable interest in the relationship between risk measures $VaR^{(n)}[X,p]$ and $VaR^{(n-1)+}[X,p]$. In particular, which risk measure gives a greater risk assessment: $VaR^{(2)}[X,p]$ — or $VaR[X_+,p]$?

The following inequality is correct.

Statement

For confidence probability values $p > F_X(0)$, $VaR^{(n)}[X,p] \geq VaR^{(n-1)+}[X,p]$.

For confidence probability values $p < F_X(0)$, $VaR^{(n)}[X,p] \leq VaR^{(n-1)+}[X,p]$.

At confidence probability $p = F_X(0)$, $VaR^{(n)}[X,p] = VaR^{(n-1)+}[X,p]$.

Proof

According to the paper [6] [6], $VaR^{(n)}[X,p] = VaR[X,1-(1-p)^n]$, and according to the formula (7), $VaR^{(n-1)+}[X,p] == VaR[X,(1-F_X(0))(1-(1-p)^{n-1})+F_X(0)]$,it is enough to compare the quantities $1-(1-p)^n$ and $(1-F_X(0))(1-(1-p)^{n-1})+F_X(0)$. Considering their difference, we get an expression: $1-(1-p)^n - (1-F_X(0))(1-(1-p)^{n-1})+F_X(0) = (1-F_X(0))(1-p)^{n-1}-(1-p)^n = (1-p)^{n-1}(p-F_X(0))$, which is obviously not negative for $p > F_X(0)$, not positively at $p < F_X(0)$ and equals zero at $p = F_X(0)$.

Statement is proved.

Consequence

For confidence probability values $p > F_X(x)$, $VaR^{(2)}[X,p] \geq VaR^{+}[X,p]$.

For confidence probability values $p < F_X(x)$, $VaR^{(2)}[X,p] \leq VaR^{+}[X,p]$.

At confidence probability $p = F_X(x)$, $VaR^{(2)}[X,p] = VaR^{+}[X,p]$.

In the case that the value of "degree" $t \geq 1$ в $VaR^{(t)}[X,p]$ is an random real number which is represented as $t = m + \alpha$, where *m* — is its whole part, and α — its fractional part ($0 \leq \alpha < 1$), the formula

$$VaR^{(t)}[X,p] = VaR[X,1-(1-p)^m(1-\alpha p)], \quad (8)$$

proved in [6] is correct, which reduces the calculation of *VaR* risk measure to the degree *t* with the confidence probability *p*, to the calculation of the normal *VaR* risk measure with a modified confidence probability $1-(1-p)^m(1-\alpha p)$.

Then using formulas (8) and (5) we get: $VaR^{(t)+}[X,p] = VaR^{+}[X,1-(1-p)^m(1-\alpha p)] = VaR[X,(1-F_X(0))(1-(1-p)^m(1-\alpha p))+F_X(0)]$, or

$$VaR^{(t)+}[X,p] = VaR[X,(1-F_X(0))(1-(1-p)^m(1-\alpha p))+F_X(0)]. \quad (9)$$

Thus, to estimate the conditional risk measure of *VaR* to the degree *t* for the value of the loss represented by the random value *X* under the condition $X \geqslant 0$ with the confidence probability *p*, it is sufficient to estimate the risk measure of *VaR* for the value of the loss, represented by a random quantity *X* with a confidence probability $(1-F_X(0))(1-(1-p)^m(1-\alpha p))+F_X(0)]$.

Note. It follows from formulas (8) and (9) that for any $p \in (0,1]$ equitable inequality: $VaR^{(t)}[X_+,p] \geq VaR^{(t)}[X,p]$, which follows from the obviously correct inequality $(1-F_X(0))(1-(1-p)^m(1-\alpha p))+F_X(0) \geq (1-(1-p)^m(1-\alpha p)$.

TRANSFORMATION OF RISK MEASURES VaR AND $VaR^{(t)}$ IN THE TRANSITION TO VaR^{+} AND $VaR^{+(t)}$ FOR CLASSICAL LOSS DISTRIBUTIONS

Let us move on to consider formulas for the transformation of risk measures VaR and $VaR^{(t)}$ in the case of the X-value representation of losses, to risk measures VaR^{+} and $VaR^{+(t)}$ to some commonly applied classical loss distributions.

The equal distribution of the value of losses X on the interval $[a,b]$ is characterized by the distribution function $F_X(x)=\begin{cases}0, if\ x<a\\ \dfrac{x-a}{b-a}, if\ a\le x\le b\\ 1, if\ x>b\end{cases}$.

The multiplicity of equally distributed values in the range $[a,b]$ is often indicated as $Uni[a,b]$.

As is known (see, for example, [14, 15]), if $X\in Uni[a,b]$, the formula $VaR[X,p]=pb+(1-p)a$ is correct.

We assume in this case that $a<0$, and $b>0$, the transition from X to X_{+} was not trivial.

Applying in this case formula (5) we get that $VaR^{+}[X,p]=[(1-F_X(0))p+F_X(0)]b+$ $+[1-(1-F_X(0))p-F_X(0)]a$.

Note that in our assumptions $F_X(0)=-\dfrac{a}{b-a}$, then $VaR^{+}[X,p]=[\dfrac{b}{b-a}p-\dfrac{a}{b-a}]b+$

$$+[\frac{b}{b-a}-\frac{b}{b-a}p]a=\frac{b^2}{b-a}p-\frac{ab}{b-a}p=\frac{b(b-a)}{b-a}p=bp.$$

i.e. $VaR^{+}[X,p]=bp$.

Let us proceed to obtaining the formula for the conversion of the risk measure $VaR^{(n)}[X,p]$ at $X\in Uni[a,b]$. Using formula (7) and expression for $VaR[X,p]$, we get

$$VaR^{(n)+}[X,p]=[(1-F_X(0))(1-(1-p)^n)+F_X(0)]b+$$

$$+[1-(1-F_X(0))(1-(1-p)^n)-F_X(0)]a=$$

$$=[(1+\frac{a}{b-a})(1-(1-p)^n)-\frac{a}{b-a}]b+[1+\frac{a}{b-a}-(1+\frac{a}{b-a})(1-(1-p)^n)]a=$$

$$=[\frac{b}{b-a}(1-(1-p)^n)-\frac{a}{b-a}]b+[\frac{b}{b-a}(1-(1-(1-p)^n))a.$$

From this formula, it is seen that at $p\to 1$, $VaR^{(n)+}[X,p]$ with a high speed, it moves to b, and the speed of the quest increases with the rise of degree n.

Accordingly, the formula for $VaR^{(t)+}[X,p]$ with a random real value $t\ge 1, t=m+\alpha$ takes the form:

$$VaR^{(t)+}[X,p]=[\frac{b}{b-a}(1-(1-p)^m(1-\alpha p))-\frac{a}{b-a}]b+[\frac{b}{b-a}(1-(1-(1-p)^m(1-.\alpha p)))a.$$

Localized indicative distribution of the loss value of X with parameters r and a, where $r>0$, $a\in R$ is characterized by the distribution function $F_X(x)=\begin{cases}1-e^{-r(x-a)}, if\ x\ge a\\ 0, if\ x<a\end{cases}$.

The multiplicity of random values distributed in the interval $[a,+\infty)$ according to this law is often referred to as $Exp(r,a)$.

It is easy to prove that if $X\in Exp(r,a)$, the formula is true (see *Appendix 1*) $VaR[X,p]=a-\ln(1-p)/r$.

Applying the formula (5) in this case, we get that $VaR^{+}[X,p] = a - \ln(1-(1-F_X(0))p - F_X(0))/r$.

Let's note that in our case $F_X(0) = \begin{cases} 1-e^{ra}, \ if\ a \le 0 \\ 0, \ if\ \ a > 0 \end{cases}$, then:

1) if the value of demolition $a \le 0$, we get

$$VaR^{+}[X,p] = a - \frac{\ln(1-e^{ra}p - 1 + e^{ra})}{r} = a - \frac{ra + \ln(1-p)}{r} = -\frac{\ln(1-p)}{r},$$

i.e. $VaR^{+}[X,p] = -\frac{\ln(1-p)}{r}$, in the process of conversion, the value of *VaR* increased by a value equal $-a \ge 0$, and became equal to the case when there is no demolition;

2) if the value of demolition $a > 0$, we get $VaR^{+}[X,p] = VaR[X,p] = a - \frac{\ln(1-p)}{r}$, i.e. in this case, the conversion of the random value does not change the value of the risk measure *VaR*, which corresponds to the intuitive view of the situation.

We got the next formula:

$$VaR^{+}[X,p] = \begin{cases} a - \frac{\ln(1-p)}{r}, if\ \ a > 0 \\ -\frac{\ln(1-p)}{r}, if\ \ a \le 0 \end{cases}.$$

Let's go to getting the formula of transformation of the risk measure $VaR^{(n)}[X,p]$ for $X \in Exp(r,a)$. Using formula (7) and expression for $VaR[X,p]$, we get:

1. if the value of demolition $\alpha \le 0$, we get

$$VaR^{(n)+}[X,p] = a - \frac{\ln(1-[(1-F_X(0))(1-(1-p)^n) + F_X(0)])}{r} =$$

$$= a - \frac{\ln(1-[e^{ra}(1-(1-p)^n) + 1 - e^{ra}])}{r} = a - \frac{\ln(e^{ra}(1-p)^n)}{r} = = -\frac{n\ln(1-p)}{r}.$$

2. if the value of demolition $a > 0$, we get

$$VaR^{(n)+}[X,p] = a - \frac{\ln(1-[(1-F_X(0))(1-(1-p)^n) + F_X(0)])}{r} = a - \frac{\ln((1-p)^n)}{r} = = a - \frac{n\ln(1-p)}{r}.$$

We got the next formula:

$$VaR^{(n)+}[X,p] = \begin{cases} a - \frac{n\ln(1-p)}{r}, if\ \ a > 0 \\ -\frac{n\ln(1-p)}{r}, if\ \ a \le 0 \end{cases}.$$

From this formula, we can see that $p \to 1$, $VaR^{(n)+}[X,p]$ linear by *n* is aimed at $+\infty$.

Accordingly, the formula for $VaR^{(t)+}[X,p]$ with a random true value $t \ge 1, t = m + \alpha$ takes the form:

$$VaR^{(t)+}[X,p] = \begin{cases} a - \frac{\ln(1-p)^m(1-\alpha p)}{r}, if\ \ a > 0 \\ -\frac{\ln(1-p)^m(1-\alpha p)}{r}, if\ a \le 0 \end{cases}.$$

Triangular distribution of the value of losses X with the summits of the base of the triangle in points a,b and mode k, where $a,b,k\in R$ and $a\le k\le b$ are characterized by the distribution function

$$F_X(x)=\begin{cases}0, if\ x\le a\\ \frac{(x-a)^2}{(b-a)(k-a)}, if\ a<x\le k\\ 1-\frac{(b-x)^2}{(b-a)(b-k)}, if\ k<x\le b\\ 1, if\ x>b\end{cases}.$$

The multiplicity of triangularly distributed values in the interval $[a,b]$ will be indicated by $Tr(a,b,k)$.

As you know (see [14]), if the value of losses is a random value $X\in Tr(a,b,k)$, then the formula is correct

$$VaR[X,p]=\begin{cases}b-\sqrt{(1-p)(b-a)(b-k)}, if\ k\le a(1-p)+bp\\ a+\sqrt{p(b-a)(k-a)}, if\ k>a(1-p)+bp\end{cases}.$$

Applying the formula (5) in this case, we get that

$$VaR^+[X,p]=\begin{cases}b-\sqrt{[1-((1-F_X(0))p+F_X(0))](b-a)(b-k)},\\ if\ k\le a(1-((1-F_X(0))p+F_X(0)))+b((1-F_X(0))p+F_X(0))\\ a+\sqrt{((1-F_X(0))p+F_X(0))(b-a)(k-a)},\\ if\ k>a(1-((1-F_X(0))p+F_X(0))+b((1-F_X(0))p+F_X(0))\end{cases}$$

Let's note that in our case $F_X(0)=\begin{cases}0, if\ a\ge 0\\ \frac{a^2}{(b-a)(k-a)}, if\ a<0\le k\\ 1-\frac{b^2}{(b-a)(b-\nu)}, if\ k<0\le b\\ 1, if\ b<0\end{cases},$

then:

1) if $a\ge 0$, we get

$$VaR^+[X,p]=\begin{cases}b-\sqrt{(1-p)(b-a)(b-k)}, if\ k\le a(1-p)+bp\\ a+\sqrt{p(b-a)(k-a)}, if\ k>a(1-p)+bp,\end{cases}$$

i.e. in this case $VaR^+[X,p]=VaR[X,p]$, which corresponds to our intuitive view of the transformation process;

2) if $a<0\le k$, we get

$$VaR^{+}[X,p]=\begin{cases} b-\sqrt{[1-((1-\frac{a^2}{(b-a)(k-a)})p+\frac{a^2}{(b-a)(k-a)})](b-a)(b-k)}, \\ if\ k\le a(1-((1-\frac{a^2}{(b-a)(k-a)})p+\frac{a^2}{(b-a)(k-a)}))+ \\ +b((1-\frac{a^2}{(b-a)(k-a)})p+\frac{a^2}{(b-a)(k-a)}); \\ a+\sqrt{((1-\frac{a^2}{(b-a)(k-a)})p+\frac{a^2}{(b-a)(k-a)})(b-a)(k-a)}, \\ if\ k> a(1-((1-\frac{a^2}{(b-a)(k-a)})p+\frac{a^2}{(b-a)(k-a)}))+ \\ +b((1-\frac{a^2}{(b-a)(k-a)})p+\frac{a^2}{(b-a)(k-a)})) \end{cases}$$

3) if $k<0\le b$, we get

$$VaR^{+}[X,p]=\begin{cases} b-\sqrt{[\frac{b^2}{(b-a)(b-k)}-\frac{b^2}{(b-a)(b-k)}p](b-a)(b-k)}, \\ if\ k\le a(\frac{b^2}{(b-a)(b-k)}-\frac{b^2}{(b-a)(b-k)}p)+ \\ +b(\frac{b^2}{(b-a)(b-k)}p+1-\frac{b^2}{(b-a)(b-k)}) \\ a+\sqrt{(\frac{b^2}{(b-a)(b-k)}p+1-\frac{b^2}{(b-a)(b-k)})(b-a)(k-a)}, \\ if\ k> a(\frac{b^2}{(b-a)(b-k)}-\frac{b^2}{(b-a)(b-k)}p)+ \\ +b(\frac{b^2}{(b-a)(b-k)}p+1-\frac{b^2}{(b-a)(b-k)}) \end{cases}$$

or

$$VaR^{+}[X,p]=\begin{cases} b-b\sqrt{1-p}, \\ if\ k\le a\frac{b^2(1-p)}{(b-a)(b-k)}+ \\ +b(1-\frac{b^2(1-p)}{(b-a)(b-k)}) \\ a+\sqrt{(1-\frac{b^2(1-p)}{(b-a)(b-k)})(b-a)(k-a)}, \\ if\ k> a\frac{b^2(1-p)}{(b-a)(b-k)}+ \\ +b(1-\frac{b^2(1-p)}{(b-a)(b-k)}) \end{cases}$$

4) if $b<0$, we get $VaR^{+}[X,p]=b$,

i.e. in this case, with any credible probability of positive values of risk (loss) does not occur.

The formulas for the conversion of risk measures $VaR^{(n)}[X,p]$ and $VaR^{(t)}[X,p]$ for $X\in Tr(a,b,k)$ the transition from random value *X* to value X_{+} are easy to write using the formulas (7) and (9) of the expressions for $VaR^{(n)}[X,p]$ and $VaR^{(t)}[X,p]$ for triangular distribution given in [6]. Because of their size, we do not bring them here.

Normal distribution of the value of losses *X* with parameters a and s (a — value of expected losses, σ — standard deviation of the losses), where $s>0$, $a\in R$ is characterized by the distribution function

$$F_X(x)=\frac{1}{\sqrt{2\pi}s}\int_{-\infty}^{x}\exp(-\frac{(t-a)^2}{2s^2})dt\,.$$

The multiplicity of random values distributed in the interval $(-\infty,+\infty)$, according to this law is often referred as $Nor(a,s)$.

As is known (see, for example, [8]), if $X\in Nor(a,s)$, then the formula is correct $VaR[X,p]=a+k_p s$, where k_p — standard normal distribution (with parameters $a=0$ and $s=1$).

Applying the formula (5) in this case, we get that

$$VaR^{+}[X,p]=VaR[X,(1-F_X(0))p+F_X(0)],$$
$$VaR^{+}[X,p]=a+k_{(1-F_X(0))p+F_X(0)}s\,.$$

Let's note that in our case $F_X(0)=\frac{1}{2}$, then $VaR^{+}[X,p]=a+k_{0,5p+0,5}s$.

Go to obtaining the risk measure recirculation formula $VaR^{(n)}[X,p]$ для $X\in Nor(a,s)$.

Using formula (7) and expression for $VaR[X,p]$, we get:

$$VaR^{(n)+}[X,p]=a+k_{0,5(1-(1-p)^n)+0,5}s\,.$$

From this formula you can see that $p\to 1$ $VaR^{(n)+}[X,p]$ id aim at $+\infty$ with a speed that increases with the growth of *n*.

Accordingly, the formula for $VaR^{(t)+}[X,p]$ with a random correct value $t\geq 1, t=m+\alpha$ takes the form:

$$VaR^{(t)+}[X,p]=a+k_{0,5(1-(1-p)^m(1-\alpha p))+0,5}s\,.$$

RISK MEASURES ES AND $ES^{(t)}$ AND REVIEWING THEIR EVALUATIONS IN THE TRANSITION TO RISK MEASURES ES^{+} AND $ES^{(t)+}$

Expectation distortion risk measures are often used to calculate economic capital. One of the frequently used distortion risk measures is the *ES* expected deficit measure. For example, the Basel Committee on Banking Supervision establishes an *ES* risk measure at a probability of 97.5% for the calculation of minimum capital requirements [16].

This measurement is applied both to assess the risks associated with the random amount of losses *X*, and to assume that the losses of *X* are non-negative ($X\geq 0$).

It is natural to introduce the following definition of a conditional risk measure $X\geq 0$ subject to an expected deficit with a given probability *p*, as follows:

$$ES^{+}[X,p]=E[X\mid X>VaR^{+}[X,p]]=E[X\mid X>(F_X^{+})^{-1}(p)],$$

where $E[X\mid A]$, indicates the conditional expectation of the random value *X* on condition of the performance of event *A*.

We will try to understand the relationship between risk measures $ES[X,p]$ and $ES^{+}[X,p]$.

Note that according to the definition of the risk measurement *ES* for the value *X* with the probability *p*, $ES[X,p]$, the ratio is correct:

$$ES[X,p]=E[X\mid X>VaR[X,p]]=E[X\mid X>F_X^{-1}(p)],\ p\in(0,1]. \qquad (10)$$

From the last ratio and definition $VaR[X,p]$ the expression follows

$$ES[X,p]=\frac{1}{1-p}\int_{F_X^{-1}(p)}^{+\infty} xf_X(x)dx, \qquad (11)$$

from which you can extract (see, for example, [8]) a well-known formula:

$$ES[X,p]=\frac{1}{1-p}\int_p^1 F_X^{-1}(q)dq=\frac{1}{1-p}\int_p^1 VaR[X,q]dq. \qquad (12)$$

According to the risk measure $ES^{+}[X,p]$ the ratio is correct:

$$ES^{+}[X,p]=E[X\mid X>VaR^{+}[X,p]]=E[X\mid X>(F_X^{+})^{-1}(p)],\ p\in(0,1]. \qquad (13)$$

Or using formulas (5) for $VaR^{+}[X,p]$ we get

$$ES^{+}[X,p]=E[X\mid X>VaR[X,(1-F_X(0))p+F_X(0)]]=$$
$$=E[X\mid X>F_X^{-1}((1-F_X(0))p)+F_X(0)].$$

Then similarly to the formula (11) we get:

$$ES^{+}[X,p]=\frac{1}{1-((1-F_X(0))p+F_X(0))}\int_{F_X^{-1}((1-F_X(0))p+F_X(0))}^{+\infty} xf_X(x)dx. \qquad (14)$$

Complete in the last integral the next replacement of the variable:

$$x=F_X^{-1}((1-F_X(0))q+F_X(0)).$$

Note that at $q=p\ \ x=F_X^{-1}((1-F_X(0))p+F_X(0)$, and at $q=1\ \ x=F_X^{-1}(1)=+\infty$.

From the theorem of the derivative reverse function, it follows that

$$dx=\frac{1}{F_X'(F_X^{-1}((1-F_X(0))q+F_X(0)))}(1-F_X(0))dq=\frac{1}{f_X(F_X^{-1}((1-F_X(0))q+F_X(0)))}(1-F_X(0))dq.$$

Therefore, from (14) follows:

$$ES^{+}[X,p]=\frac{1}{1-((1-F_X(0))p+F_X(0))}\times$$
$$\times\int_p^1 F_X^{-1}((1-F_X(0))q+F_X(0))f_X(F_X^{-1}((1-F_X(0))q+F_X(0)))\frac{1-F_X(0)}{f_X(F_X^{-1}((1-F_X(0))q+F_X(0)))}dq= \qquad (15)$$
$$=\frac{1-F_X(0)}{1-((1-F_X(0))p+F_X(0))}\int_p^1 F_X^{-1}((1-F_X(0))q+F_X(0))dq.$$

Replacing another variable in the last integral: $r=(1-F_X(0))q+F_X(0)$.

Note that at $q=p$, $r=(1-F_X(0))p+F_X(0)$, and at $q=1$ $r=1$ and $dr=(1-F_X(0))dq$, and means,

$$dq=\frac{1}{1-F_X(0)}dr.$$

Therefore, from (15) we get

$$ES^{+}[X,p]=\frac{1}{1-((1-F_X(0))p+F_X(0))}\int_{(1-F_X(0))p+F_X(0)}^{1}F_X^{-1}(r)dr,$$

of which, according to the formula (12) and we get the following important equation:

$$ES^{+}[X,p]=ES[X,(1-F_X(0))p+F_X(0)]. \tag{16}$$

Thus, in order to estimate the *ES* risk measure for the value of losses represented by the random value *X*, provided that $X\geq 0$, with confidence probability *p*, it is sufficient to assess the risk measured by *ES* for the size of the losses presented by a random value *X* with confidence probability $(1-F_X(0))p+F_X(0)$.

Note. Obviously, for any $p\in(0,1]$ the inequality is correct:

$$ES[X_{+},p]\geq ES[X,p].$$

In the papers [6] and [7] examined the family of disaster risk measures "*ES* in degree *t*", $ES^{(t)}[X,p]$, which proved to be a subset of well-examined risk measurements considered in a large number of works (see [8–13]) of risk distortion expectation measures.

Let's examine how risk assessments measured by catastrophic risk measurements $ES^{(t)+}[X,p]$, relate to risk measures $ES^{(t)}[X,p]$.

Let's start with the case when the value of the "degree" *t* in $ES^{(t)}[X,p]$ is a natural number: $t=n$.

Let us note that in this case, as demonstrated in the paper [6], the correct formula

$$ES^{(n)}[X,p]=ES[X,1-(1-p)^{n}], \tag{17}$$

which calculates the risk measurement of *ES* n degree *n* with probability *p*, to the calculation of the normal risk of *ES* with modified probability $1-(1-p)^{n}$.

Then, using the formulas (17) and (16) applied to the random value X_{+}, we get

$$ES^{(n)+}[X,p]=ES^{+}[X,1-(1-p)^{n}]=ES[X,(1-F_X(0))(1-(1-p)^{n})+F_X(0)],$$

или

$$ES^{(n)+}[X,p]=ES[X,(1-F_X(0))(1-(1-p)^{n})+F_X(0)]. \tag{18}$$

Thus, in order to assess the risk measurement of *ES* in degree *n*, assuming that the loss value takes only non-negative values $X\geq 0$, with confidence probability *p* is sufficient to estimate the risk of *ES* for the value of losses represented by the random value *X* with confidence probability $(1-F_X(0))(1-(1-p)^{n})+F_X(0)$.

Note. It follows from the formulas (17) and (18) that in any $p\in(0,1]$ is correct inequality: $ES^{(n)+}[X,p]\geq ES^{(n)}[X,p]$.

There is considerable interest in the ratio between risk measures $ES^{(n)}[X,p]$ and $ES^{(n-1)+}[X,p]$. In particular, which risk measure gives a greater risk assessment $ES^{(2)}[X,p]$ or $ES[X_+,p]$?

It turns out that the following statement is correct:

Statement

1. If $p > F_X(0)$, then $ES^{(n)+}[X,p] \le ES^{(n+1)}[X,p]$.
2. If $p < F_X(0)$, then $ES^{(n)+}[X,p] \ge ES^{(n+1)}[X,p]$.
3. If $p = F_X(0)$, then $ES^{(n)+}[X,p] = ES^{(n)}[X,p]$.

Proof

Remember that according to the formula (18)

$$ES^{(n)+}[X,p] = ES[X,(1-F_X(0))(1-(1-p)^n)+F_X(0)],$$

then

$$ES^{(n+1)}[X,p] = ES[X,1-(1-p)^{n+1}].$$

1. When $p > F_X(0)$, $1-p < 1-F_X(0)$, i.e. $(1-p)^{n+1} < (1-p)^n(1-F_X(0))$, then $1-F_X(0)-(1-p)^n(1-F_X(0))+F_X(0) < 1-(1-p)^{n+1}$,

i.e.

$$(1-F_X(0))(1-(1-p)^n)+F_X(0) < 1-(1-p)^{n+1}.$$

Then $ES^{(n)+}[X,p] \le ES^{(n+1)}[X,p]$.

2. When $p < F_X(0)$, $1-p > 1-F_X(0)$, i.e. $(1-p)^{n+1} > (1-p)^n(1-F_X(0))$, then $1-F_X(0)-(1-p)^n(1-F_X(0))+F_X(0) > 1-(1-p)^{n+1}$,

i.e.

$$(1-F_X(0))(1-(1-p)^n)+F_X(0) > 1-(1-p)^{n+1}.$$

Then $ES^{(n)+}[X,p] \ge ES^{(n+1)}[X,p]$.

3. When $p = F_X(0)$

$$(1-F_X(0))(1-(1-p)^n)+F_X(0) = (1-p)(1-(1-p)^n)+p = 1-(1-p)^{n+1}.$$

Then $ES^{(n)+}[X,p] = ES^{(n+1)}[X,p]$.

Then, consequence is correct:

Consequence

1. If $p > F_X(0)$, then $ES^{+}[X,p] \le ES^{(2)}[X,p]$.
2. If $p < F_X(0)$, then $ES^{+}[X,p] \ge ES^{(2)}[X,p]$.
3. If $p = F_X(0)$, then $ES^{+}[X,p] = ES^{(2)}[X,p]$.

As is obvious (see [15]), without knowing the law of the distribution of losses, it is impossible to say which of the amounts of risk $ES[X,p]$ or $VaR^{(2)}[X,p]$ more.

Furthermore, as demonstrated in the work [17], a measure $ES[X,p]$ is multiplied by a certain mix of risk measures $VaR^{(2)}[X,p]$ $ES^{(2)}[X,p]$, namely, equal inequality $ES[X,p] \le pVaR^{(2)}[X,p] + (1-p)ES^{(2)}[X,p]$ for any confidence probability *p*, i.e. the value of the risk measure $ES[X,p]$ does not exceed the probability-weighted *p* and 1 — *p* sum of risks measures $VaR^{(2)}[X,p]$ and $ES^{(2)}[X,p]$.

Then, obviously, the next series of inequalities is correct:

$$ES[X,p] \le ES^{+}[X,p] \le pVaR^{(2)+}[X,p] + (1-p)ES^{(2)+}[X,p],$$

enabling a more rigorous assessment of catastrophic risks through the conversion procedure, using a suitable mix of risk measures $VaR^{(2)+}[X,p]$ and $ES^{(2)+}[X,p]$.

For instance, when the value of "degree" $t \geq 1$ in $ES^{(t)}[X,p]$ is a random substantial number, which is presented in the form $t = m + \alpha$, where m — is its whole part, and α — is its fractional part ($0 \leq \alpha < 1$),as demonstrated in work [6], the formula is correct

$$ES^{(t)}[X,p] = ES[X, 1-(1-p)^m(1-\alpha p)], \tag{19}$$

which calculates the risk measurement *ES* in degree *t* with probability *p*, to the calculation of the normal risk measure *ES* with modified probability $1-(1-p)^m(1-\alpha p)\,\alpha p$.

Then, using the formulas (16) and (19) applied in this case, we get: $ES^{(t)+}[X,p] = ES^{+}[X, 1-(1-p)^m(1-\alpha p)] = ES[X,(1-F_X(0))(1-(1-p)^m(1-\alpha p))+F_X(0)]$,

or

$$ES^{(t)+}[X,p] = ES[X,(1-F_X(0))(1-(1-p)^m(1-\alpha p))+F_X(0)]. \tag{20}$$

Thus, for the assessment of the risk measurement *ES* in degree *t* for the value of losses represented by the random value *X*, provided that $X \geq 0$, with confidence probability *p*, it is sufficient to estimate the risk measured by *ES* for the size of the losses presented by a random value *X* with confidence probability $(1-F_X(0))(1-(1-p)^m(1-\alpha p))+F_X(0)$.

Note. It follows from the formulas (8) and (9) that for any $p \in (0,1]$ is correct inequality: $VaR^{(t)+}[X,p] \geq VaR^{(t)}[X,p]$.

TRANSFORMATION OF RISK MEASURE ES AND $ES^{(t)}$ AND TRANSITION TO RISK MEASURE ES^{+} AND $ES^{(t)+}$ TO CLASSICAL LOSS DISTRIBUTIONS

Consider the formula for the conversion of risk measures ES and $ES^{(t)}$ to appropriate risk conditional measures, provided that $X \geq 0$ for some commonly used classical loss distributions.

Equal distribution of the value of losses *X* on the interval $[a,b]$.

As you know (see, for example [15]), if $X \in Uni[a,b]$, then the formula is correct $ES[X,p] = \dfrac{(1-p)a+(1+p)b}{2}$.

In this case, we assume that $a < 0$, and $b > 0$, the transition is not trivial.

In this case, using the formula (16), we get:

$$ES^{+}[X,p] = \frac{1}{1-F_X(0)} \frac{(1-((1-F_X(0))p+F_X(0)))a+(1+((1+F_X(0))p+F_X(0)))b}{2}.$$

Note that in our assumptions $F_X(0) = -\dfrac{a}{b-a}$, then

$$(1-F_X(0))p+F_X(0) = (1+\frac{a}{b-a)})p - \frac{a}{b-a} = \frac{b}{b-a}p - \frac{a}{b-a}, \text{ and means}$$

$$ES^{+}[X,p] = \frac{[1-\frac{b}{b-a}p+\frac{a}{b-a}]a+[1+\frac{b}{b-a}p-\frac{a}{b-a}]b}{2} = \frac{a(1+\frac{a-bp}{b-a})+b(1-\frac{a-bp}{b-a})}{2}.$$

I.e.

$$ES^{+}[X,p]=\frac{a(1+\frac{a-bp}{b-a})+b(1-\frac{a-bp}{b-a})}{2}.$$

Note, that $p\to 1$ $ES^{+}[X,p]\to b$.

Let us proceed to obtaining the appropriate formula for the transformation of the risk measure $ES^{(n)}[X,p]$ for $X\in Uni[a,b]$. Using the formula (19) and the expression for $ES[X,p]$, we get

$$ES^{(n)+}[X,p]=ES^{+}[X,1-(1-p)^{n}]=[a(1+\frac{a-b(1-(1-p)^{n})}{b-a})+b(1-\frac{a-b(1-(1-p)^{n})}{b-a})]/2.$$

Accordingly, the formula for $ES^{(t)+}[X,p]$ with a random correct value $t\geq 1, t=m+\alpha$ takes the form:

$$ES^{(t)}[X_{+},p]=\frac{1}{2}[a(1+\frac{a-b(1-(1-p)^{m}(1-\alpha p))}{b-a})+b(1-\frac{a-b(1-(1-p)^{m}(1-\alpha p))}{b-a})].$$

Variable exponential distribution of the value of losses *X* with parameters r and a, where $r>0,\ a\in R$.

It can be proved that if $X\in Exp(r,a)$, the formula is correct (see *Appendix 2*)

$$ES[X,p]=a+\frac{1}{r}-\frac{\ln(1-p)}{r}.$$

Applying in this case the formula (16) we get that

$$ES^{+}[X,p]=a+\frac{1}{r}-\frac{\ln(1-(1-F_{X}(0))p-F_{X}(0))}{r}.$$

Note that in our case $F_{X}(0)=\begin{cases}1-e^{ra}, if\ a\leq 0\\ 0, if\ a>0\end{cases}$, then:

1) if the value of the demolition $a\leq 0$, then $(1-F_{X}(0))p+F_{X}(0)=e^{ra}p+1-e^{ra}$,

and we get

$$ES^{+}[X,p]=\alpha+\frac{1}{r}-\frac{\ln(e^{ra}-e^{ra}p)}{r}=\alpha+\frac{1}{r}-\frac{ra+\ln(1-p)}{r}=$$

$$=\frac{1}{r}(1-\ln(1-p)),\ \text{i.e. } ES^{+}[X,p]=\frac{1}{r}(1-\ln(1-p));$$

2) if the value of the demolition $\alpha>0$, then $(1-F_{X}(0))p+F_{X}(0)=p$,

we get $ES^{+}[X,p]=ES[X,p]=\alpha+\frac{1}{\lambda}-\frac{\ln(1-p)}{\lambda}$, i.e. in this case, the transformation of the random value does not change the value of the risk measure ES, which corresponds to the intuitive view of the situation.

Thus, we have received the following formula:

$$ES^{+}[X,p]=\begin{cases}a+\frac{1}{r}-\frac{\ln(1-p)}{r}, if\ a>0\\ \frac{1}{r}(1-\ln(1-p)), if\ a\leq 0\end{cases}.$$

Let us proceed to obtaining the formula for the conversion of the risk measure $ES^{(n)}[X,p]$ for $X \in Exp(r,a)$. Using the last expression and formula for $ES^{(n)}[X,p]$, we get:

$$ES^{(n)+}[X,p]=\begin{cases} a+\frac{1}{r}-\frac{\ln(1-(1-(1-p)^n)}{r}, if\ a>0 \\ \frac{1}{r}(1-\ln(1-(1-(1-p)^n)), if\ a\le 0 \end{cases},$$

and means

$$ES^{(n)+}[X,p]=\begin{cases} \alpha+\frac{1}{r}-\frac{n\ln(1-p)}{r}, if\ a>0 \\ \frac{1}{r}(1-n\ln(1-p)), if\ a\le 0 \end{cases}$$

From this formula, we can see that $p \to 1$ $ES^{(n)+}[X,p]$ linear by n is aimed at $+\infty$.
Accordingly, the formula for $ES^{(t)+}[X,p]$ with a random correct value $t \ge 1$, $t=m+\alpha$ takes the form:

$$ES^{(t)+}[X,p]=\begin{cases} a+\frac{1}{r}-\frac{\ln[(1-p)^m(1-\alpha p)]}{r}, if\ \ a>0 \\ \frac{1}{r}(1-\ln[(1-p)^m(1-\alpha p)], if\ \ a\le 0 \end{cases}.$$

Triangular distribution of the X loss value with the tops of the base of the triangle in points a,b and mode k, where $a,b,k \in R$ and $a \le k \le b$.
We can argue that if $X \in Tr(a,b,k)$, the formula is correct (see *Appendix 3*)

$$ES[X,p]=\begin{cases} b-\frac{2}{3}\sqrt{(1-p)(b-a)(b-k)}, if\ p\ge\frac{k-a}{b-a} \\ \frac{1}{1-p}[b+a-k+\frac{2}{3}\sqrt{(b-a)(k-a)}(\sqrt{(\frac{k-a}{b-a})^3}-\sqrt{(\frac{b-k}{b-a})^3}-\sqrt{p^3})], if\ p<\frac{k-a}{b-a} \end{cases}.$$

Applying the formula (16) in this case, we get that

$$ES^{+}[X,p]=\begin{cases} b-\frac{2}{3}\sqrt{(1-((1-F_X(0))p+F_X(0))(b-a)(b-k)}, \\ if\ p\ge(\frac{k-a}{b-a}-F_X(0))/(1-F_X(0)) \\ \frac{1}{1-p}[b+a-k+\frac{2}{3}\sqrt{(b-a)(k-a)}\times \\ \times(\sqrt{(\frac{k-a}{b-a})^3}-\sqrt{(\frac{b-k}{b-a})^3}-\sqrt{((1-F_X(0)p+F_X(0))^3})], \\ if\ p<(\frac{k-a}{b-a}-F_X(0))/(1-F_X(0)) \end{cases}.$$

Note that in our case $F_X(0)=\begin{cases}0, if\ a\geq 0\\ \frac{a^2}{(b-a)(k-a)}, if\ a<0\leq k\\ 1-\frac{b^2}{(b-a)(b-k)}, if\ k<0\leq b\\ 1, if\ b<0\end{cases}$,

and:

1) if $a\geq 0$, then
2)

$$ES^{+}[X,p]=\begin{cases}b-\frac{2}{3}\sqrt{(1-p)(b-a)(b-k)}, if\ p\geq\frac{k-a}{b-a}\\ \frac{1}{1-p}[b+a-k+\frac{2}{3}\sqrt{(b-a)(k-a)}(\sqrt{(\frac{k-a}{b-a})^3}-\sqrt{(\frac{b-k}{b-a})^3}-\sqrt{p^3})], if\ p<\frac{k-a}{b-a}\end{cases},$$

i.e. in this case в этом случае $ES^{+}[X,p]=ES[X,p]$, which corresponds to our intuitive view of the transformation process;

3) if $a<0\leq k$, and

$$(\frac{k-a}{b-a}-F_X(0))/(1-F_X(0))=(\frac{k-a}{b-a}-\frac{a^2}{(b-a)(k-a)})/(1-\frac{a^2}{(b-a)(k-a)})=$$

$$=\frac{k(k-2a)}{k(b-a)-ab},$$

we get $$ES^{+}[X,p]=\begin{cases}b-\frac{2}{3}\sqrt{[1-((1-\frac{a^2}{(b-a)(k-a)})p+\frac{a^2}{(b-a)(k-a)})](b-a)(b-k)},\\ if\ p\geq\frac{k(k-a)}{k(b-a)-ab};\\ \frac{1}{1-p}[b+a-k-ap+\frac{2}{3}\sqrt{(b-a)(k-a)}(\sqrt{(\frac{k-a}{b-a})^3}-\sqrt{(\frac{b-k}{b-a})^3}-\\ -\sqrt{((1-\frac{a^2}{(b-a)(b-k)})p-\frac{a^2}{(b-a)(b-k)})^3})]\\ if\ p<\frac{k(k-2a)}{k(b-a)-ab}\end{cases};$$

4) if $k<0\leq b$, we get

$$ES^{+}[X,p]=\begin{cases} b-\dfrac{2}{3}\sqrt{[\dfrac{b^2}{(b-a)(b-k)}-\dfrac{b^2}{(b-a)(b-k)}p](b-a)(b-k)}, \\ if\ p\geq\dfrac{k(k-2a)}{k(b-a)-ab}; \\ \dfrac{1}{1-p}[b+a-k+\dfrac{2}{3}\sqrt{(b-a)(k-a)}\times \\ \times(\sqrt{(\dfrac{k-a}{b-a})^3}-\sqrt{(\dfrac{b-k}{b-a})^3}-\sqrt{(\dfrac{b^2}{(b-a)(b-k)}p+1-\dfrac{b^2}{(b-a)(b-k)})^3})] \\ if\ p<\dfrac{k(k-2a)}{k(b-a)-ab} \end{cases},$$

or

$$ES^{+}[X,p]=\begin{cases} b-\dfrac{2}{3}b\sqrt{1-p}, \\ if\ p\geq\dfrac{k(k-2a)}{k(b-a)-ab}; \\ \dfrac{1}{1-p}[b+a-k+\dfrac{2}{3}\sqrt{(b-a)(k-a)}\times \\ \times(\sqrt{(\dfrac{k-a}{b-a})^3}-\sqrt{(\dfrac{b-k}{b-a})^3}-\sqrt{(1-\dfrac{b^2(1-p)}{(b-a)(b-k)})^3})] \\ if\ p<\dfrac{k(k-2a)}{k(b-a)-ab} \end{cases};$$

5) if $b<0$, we get

$$ES^{+}[X,p]=b,$$

i.e. in this case, with any credible probability of positive values of risk (loss) does not occur.

The formulas for the conversion of risk measures $ES^{(n)}[X,p]$ and $ES^{(t)}[X,p]$ for $X\in Tr(a,b,\nu)$ is easy to write using the formulas (18) and (20) the expressions for $ES[X,p]$ in the case of triangular distribution obtained in *Appendix 3*. Because of their size, we do not bring them here.

Normal distribution of the value of losses *X* with parameters a and *s* (a — value of expected losses, *s* — is the standard deviation of the losses), where $s>0,\ a\in R$.

As is known (see, for example, [8]), if $X\in Nor(a,s)$, then the formula is correct

$$ES[X,p]=a+\frac{\exp(-0,5k_p^2)}{(1-p)\sqrt{2\pi}}s,,$$

where k_p — standard normal distribution (with parameters $a=0$ and $s=1$).

Applying the formula (16) in this case, we get that

$$ES^{+}[X,p]=a+\frac{\exp(-0,5k^2_{(1-F_X(0))p+F_X(0)})}{(1-p)\sqrt{2\pi}}s.$$

Note that in our case $F_X(0)=\frac{1}{2}$, then

$$ES^{+}[X,p]=a+\frac{\exp(-0,5k^2_{0,5p+0,5})}{(1-p)\sqrt{2\pi}}s.$$

Let us proceed to obtaining the formula for the conversion of the risk measure $ES^{(n)}[X,p]$ for $X \in Nor(a,\sigma)$ the transition to the appropriate risk conditional measure.

Using the formula (18) and the expression for $ES^{+}[X,p]$, we get:

$$ES^{(n)+}[X,p] = a + \frac{\exp(-0,5k^2_{0,5(1-(1-p)^n)+0,5})}{(1-p)\sqrt{2\pi}}s.$$

From this formula you can see that at $p \to 1$ $ES^{(n)+}[X,p]$ is aimed at $+\infty$ at a speed that increases with the growth of n.

Accordingly, the formula for $ES^{(t)+}[X,p]$ with a random correct value $t \geq 1, t = m + \alpha$ takes the form:

$$ES^{(t)+}[X,p] = a + \frac{\exp(-0,5k^2_{0,5(1-(1-p)^m(1-\alpha p))+0,5})}{(1-p)\sqrt{2\pi}}s.$$

CONCLUSION

Most often we assume that the value of losses *X* is a random value distributed throughout the line $(-\infty,+\infty)$. But sometimes researchers study the losses, assuming they are non-negative, $X \geq 0$. In this paper, we first explore how the distribution of probability of this value is transformed by such a transition. We examine the formulas of transformation of risk assessment of different catastrophicity when adding this condition. As it turns out, when the random value of losses *X*, is replaced by the value X_+, this does not affect the assessment of risk measures such as VaR, ES, $VaR^{(t)}$ and $ES^{(t)}$. However, when the random loss distribution is replaced by its conditional distribution under the $X \geq 0$, condition, both the risk assessment formulas and their assessment values are changed, sometimes significantly.

We hope that the results of this study will contribute to a better understanding of the theoretical assertions relating to risk measurements, as well as the results of practical risk assessments, depending on whether the assessment was based on allowing losses to accept negative values or focusing just on positive values.

REFERENCES

6. Borovkov A.A. Probability theory. Moscow: Nauka; 1986. 431 p. (In Russ.).
7. Santolino M., Belles-Sampera J., Sarabia J.M., Guillen M. An examination of the tail contribution to distortion risk measures. *Journal of Risk*. 2021;23(6):88–113. DOI: 10.21314/JOR.2021.014
8. Crouhy M., Galai D., Mark R. The essentials of risk management. New York, NY: McGraw-Hill Book Co.; 2005. 416 p. (Russ. ed.: Crouhy M., Galai D., Mark R. Osnovy risk-menedzhmenta. Moscow: Urait; 2018. 390 p.).
9. Hull J.C. Risk management and financial institutions. New York, NY: Pearson Education International; 2007. 576 p.
10. Jorion P. Value at risk: The new benchmark for managing financial risk. New York, NY: McGraw-Hill Education; 2007. 624 p.
11. Minasyan V.B. New ways to measure catastrophic financial risks: "VaR to the power of t" measures and how to calculate them. *Finance: Theory and Practice*. 2020;24(3):92–109. DOI: 10.26794/2587–5671–2020–24–3–92–109
12. Minasyan V.B. New risk measures "VaR to the power of t" and "ES to the power of t" and distortion risk measures. *Finance: Theory and Practice*. 2020;24(6):92–107. DOI: 10.26794/2587–5671–2020–24–6–92–107
13. Denuit M., Dhaene J., Goovaerts M., Kaas R. Actuarial theory for dependent risks: Measures, orders and models. Chichester: John Wiley & Sons, Ltd; 2005. 480 p. DOI: 10.1002/0470016450

14. Zhu L., Li H. Tail distortion risk and its asymptotic analysis. *Insurance: Mathematics and Economics*. 2012;51(1):115–121. DOI: 10.1016/j.insmatheco.2012.03.010
15. Yang F. First- and second-order asymptotics for the tail distortion risk measure of extreme risks. *Communications in Statistics — Theory and Methods*. 2015;44(3):520–532. DOI: 10.1080/03610926.2012.751116
16. Yin C., Zhu D. New class of distortion risk measures and their tail asymptotics with emphasis on Va R. *Journal of Financial Risk Management*. 2018;7(1):12–38. DOI: 10.4236/jfrm.2018.71002
17. Belles-Sampera J., Guillén M., Santolino M. Beyond value-at-risk: GlueVaR distortion risk measures. *Risk Analysis*. 2014;34(1):121–134. DOI: 10.1111/risa.12080
18. Belles-Sampera J., Guillén M., Santolino M. GlueVaR risk measures in capital allocation applications. *Insurance: Mathematics and Economics*. 2014;58:132–137. DOI: 10.1016/j.insmatheco.2014.06.014
19. Minasyan V.B. Risk assessment models of the companies implementing R&D projects. *Finance: Theory and Practice*. 2019;23(1):133–146. DOI: 10.26794/2587–5671–2019–23–1–133–146
20. Minasyan V.B. New risk measure VaR in the square and its calculation. Case of general law loss distributions, comparison with other risk measures. *Upravlenie finansovymi riskami = Financial Risk Management Journal*. 2019;(4):298–320. (In Russ.).
21. Minimum capital requirements for market risk. Basel Committee on Banking Supervision. Basel: Bank for International Settlements; 2019. 136 p. URL: https://www.bis.org/bcbs/publ/d457.pdf
22. Minasyan V.B. On comparison of certain measures of catastrophic risks. *Upravlenie finansovymi riskami = Financial Risk Management Journal*. 2022;(4):284–289. (In Russ.). DOI: 10.36627/2221–7541–2022–4–4–284–289

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APPENDICES

Appendix 1

Statement

If $X \in Exp(r,a)$, then the formula is correct $VaR[X,p] = a - \frac{\ln(1-p)}{r}$.

Proof

From the definition $VaR[X,p]$ it follows that its significance in any $p \in (0,1]$ is the solution of the equation

$F_X(x) = p$, i.e. $1 - e^{-r(x-a)} = p$, solving this equation relative to *x*, we get $x = a - \frac{\ln(1-p)}{r}$,

and therefore $VaR[X,p] = a - \frac{\ln(1-p)}{r}$.

Appendix 2

Statement

If $X \in Exp(r,a)$, then the formula is correct $ES[X,p] = a + \frac{1}{r} - \frac{\ln(1-p)}{r}$.

Proof

It follows from the definition in *Appendix 1* that:

$$VaR[X,p] = a - \frac{\ln(1-p)}{r},$$

but then $ES[X,p] = \frac{1}{1-p}\int_1^p VaR[X,q]dq$, we get: $ES[X,p] = \frac{1}{1-p}\int_1^p (a - \frac{\ln(1-q)}{r})dq$.

Applying the integration by parts, we get:

$$ES[X,p] = \frac{1}{1-p}(a - ap + \frac{1}{r}(1-q)\ln(1-q)|_p^1 + \frac{1}{r}\int_1^p dq) =$$

$$= \frac{1}{1-p}(a(1-p) + \frac{1}{r}(1-p)\ln(1-p) + \frac{1}{r}(1-p)) = a + \frac{1}{r} - \frac{\ln(1-p)}{r}, \text{ then } ES[X,p] = a + \frac{1}{r} - \frac{\ln(1-p)}{r}.$$

Appendix 3

Statement

If $X \in Tr(a,b,k)$, then the formula is correct

$$ES[X,p] = \begin{cases} b - \frac{2}{3}\sqrt{(1-p)(b-a)(b-k)}, if\ p \geq \frac{k-a}{b-a} \\ \frac{1}{1-p}[b + a - k + \frac{2}{3}\sqrt{(b-a)(k-a)}(\sqrt{(\frac{k-a}{b-a})^3} - \sqrt{(\frac{b-k}{b-a})^3} - \sqrt{p^3})], if\ p < \frac{k-a}{b-a}. \end{cases}$$

Proof

As you know (see [12]), if the value of losses is a random value $X \in Tr(a,b,k)$, then the formula is correct

$$VaR[X,p] = \begin{cases} b - \sqrt{(1-p)(b-a)(b-k)}, if\ k \leq a(1-p) + bp \\ a + \sqrt{p(b-a)(k-a)}, if\ k > a(1-p) + bp. \end{cases}$$

However, since inequality $k \leq a(1-p) + bq$ is equal to inequity $p \geq \frac{b-k}{b-a}$, and inequality $k > a(1-q) + bq$ is equivalent to inequality $p < \frac{b-k}{b-a}$, the expression for $VaR[X,p]$ can be rewritten as:

$$VaR[X,p]=\begin{cases} b-\sqrt{(1-p)(b-a)(b-k)}, if\ p\geq\dfrac{k-a}{b-a} \\ a+\sqrt{p(b-a)(k-a)}, if\ p<\dfrac{k-a}{b-a}. \end{cases}$$

As we know, the formula is correct $ES[X,p]=\frac{1}{1-p}\int_p^1 VaR[X,q]dq$.
There are two possible cases:

a) if $p\geq\frac{k-a}{b-a}$, then

$$ES[X,p]=\frac{1}{1-p}\int_p^1[b-\sqrt{(1-q)(b-a)(b-k)}]dq=$$

$$=\frac{1}{1-p}(bq+\frac{2}{3}\sqrt{(1-q)^3(b-a)(b-k)})\,|_p^1=\frac{1}{1-p}(b(1-p)-\frac{2}{3}\sqrt{(1-p)^3(b-a)(b-k)})=$$

$$=b-\frac{2}{3}\sqrt{(1-p)(b-a)(b-k)}),$$

i.e.

$$ES[X,p]=b-\frac{2}{3}\sqrt{(1-p)(b-a)(b-k)});$$

b) if $p<\frac{k-a}{b-a}$, then

$$ES[X,p]=\frac{1}{1-p}\int_p^{\frac{k-a}{b-a}}[a+\sqrt{q(b-a)(b-k)}]dq+\frac{1}{1-p}\int_{\frac{k-a}{b-a}}^1[b-\sqrt{(1-q)(b-a)(b-k)}]dq=$$

$$=\frac{1}{1-p}[aq+\frac{2}{3}\sqrt{q^3(b-a)(b-k)}]\,|_p^{\frac{k-a}{b-a}}+\frac{1}{1-p}[bq+\frac{2}{3}\sqrt{(1-q)^3(b-a)(b-k)}]\,|_{\frac{k-a}{b-a}}^1=$$

$$=\frac{1}{1-p}[a(\frac{k-a}{b-a}-p)+\frac{2}{3}\sqrt{(b-a)(b-k)}(\sqrt{(\frac{k-a}{b-a})^3}-\sqrt{p^3})+$$

$$+b(1-\frac{k-a}{b-a})-\frac{2}{3}\sqrt{(b-a)(b-k)}\sqrt{(1-\frac{k-a}{b-a})^3}]=$$

$$=\frac{1}{1-p}[b+a-k-ap+\frac{2}{3}\sqrt{(b-a)(b-k)}(\sqrt{(\frac{k-a}{b-a})^3}-\sqrt{(\frac{b-k}{b-a})^3}-\sqrt{p^3})].$$

Thus, we get:

$$ES[X,p]=\begin{cases} b-\dfrac{2}{3}\sqrt{(1-p)(b-a)(b-k)}, if\ p\geq\dfrac{k-a}{b-a} \\ \dfrac{1}{1-p}[b+a-k+\dfrac{2}{3}\sqrt{(b-a)(k-a)}(\sqrt{(\dfrac{k-a}{b-a})^3}-\sqrt{(\dfrac{b-k}{b-a})^3}-\sqrt{p^3})], if\ p<\dfrac{k-a}{b-a}. \end{cases}$$