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The Study of Stochastic Ranking Scales for Financial Risks of Different Levels of Catastrophicity

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ABSTRACT

The concept of stochastic order between random variables has been formed and developed in science since the middle of the 20th century. However, it was only in the last decade that these concepts began to penetrate into risk management applications and into the actuarial field. This has made it possible for them to become an actively developing area, allowing us to compare the risk factors themselves, as well as the level of hazards (risks) associated with them. **The purpose** of this work is to develop recommendations for solving the problem of comparing various risks and to propose methods for comparing risks with different degrees of catastrophicity. The relations of stochastic and dispersion orders in a set of risks are investigated. As **a result**, a study of these orders is carried out using risk measures VaR , as well as risk measures $VaR^{(t)}$ (VaR to the power of t), introduced into scientific use in recent years. The description of these orders is analyzed using the stop loss **method** of risk transformation, as well as the danger rate **method**. In addition, in this study, for well-known examples of loss distributions are given that find wide application in risk management and actuarial science. Simple methods are obtained to allow one to compare risks under these distributions, which are expressed through the parameters of these distributions. The results and methods presented in this work may be of interest to both practitioners and researchers involved in the problems of risk management and the actuarial fields.

Keywords: stochastic order; dispersion order; stop loss risk transformation; danger rate; risk measure VaR ; risk measure VaR to the power of t ; spreads VaR and VaR to the power of t

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INTRODUCTION

Due to their expertise in statistics, finance, and business, risk managers help companies assess risks and create policies to minimize them. They help reduce the financial impact of risks on companies and their customers.

One of the main tasks of risk managers is assessing risks and developing strategies to minimize their cost. This is why businesses need risk managers to ensure their operations are safe and profitable.

Risk comparison is at the heart of the risk management profession. Every business faces different risks, and risk managers must place these risks in order to manage them effectively.

There are two reasons why some risks may be more preferable than others. One reason is that some risks are greater in magnitude, while others have a wider range of possible outcomes. A wider range means that there is a higher probability of extreme outcomes, which can make the risk seem more unpredictable.

However, it's important to clarify what we mean by "lighter tail" and "heavier tail". We cannot expect that one risk will always have values that are less than those of another risk, as this would be impossible to predict.

Ultimately, risk managers need to carefully consider all the factors involved in each risk in order to make informed decisions about how to manage them. But from the perspective of possible significant, in a certain sense, realizable losses, one risk may be more preferable than another. The heaviness of the tail can also be measured using different metrics, and this can lead to different risk comparison scales.

When we can measure the risk associated with different random variables, our goal is to formalize the idea that one risk is more dangerous than another. We could base our comparison on a single measure of risk, but the result is too rough. Instead, we prefer to use a wide range of risk measures, which leads to different orders among random variables representing risks. These orders are called stochastic orders, and they allow us to compare probabilistic measurements.

The easiest way to rank risky situations is to calculate a measure of risk and sort random prospects according to its values. However, in many cases, even if a risk measure seems reasonable in the given context, the values of associated parameters, such as confidence probabilities and distortion functions VaR and ES , are not uniquely determined by the problem at hand. Therefore, risk managers often want more than a simple comparison based on a single risk measure. They may request that the risk be more favorable under any of a range of specified risk measures. This leads to a partial order, rather than a complete order (i.e., the risk manager no longer has an order for every pair of risks). This is known as a stochastic order.

The so-called stochastic order is undoubtedly well-known and seems to be the most intuitive stochastic order. It was introduced back in Lehmann [1] and analyzed by Kamae et al. [2]. However, in recent decades, a large number of other orders have emerged, mainly based on the motivation of various fields of application (statistics, queue theory, reliability theory, economics, biomathematics, actuarial science, physics, etc.). They generated a huge amount of literature, culminating in a bibliography in Mosler, Scarsini [3] and books by Shaked, Shantikumar [4], and Müller, Stoyan [5].

In recent decades, this interest has grown to such an extent that partial orders have become one of the most important tools for comparing the riskiness of various random situations. Interesting actuarial applications of stochastic orders are discussed in books by Goovaerts et al. [6] and Kaas et al. [7].

In this paper, we continue to study some partial orders in a variety of risks using both the well-known risk measure VaR and a relatively new family of risk measures " VaR to the power of t " (see Minasyan [8, 9]).

STOCHASTIC ORDER RELATIONSHIPS BETWEEN RISKS

Partial Orders between Distribution Functions

Most of the order relations that we consider in this paper are partial orders on the set

of distribution functions. Let's clarify this concept.

Definition 1

Let Ψ be the set of one — dimensional distribution functions of various random variables representing the risk factors we are interested in. The binary relation, which we will denote by $(\leq)$, is a partial order on Ψ if for any elements of the distribution functions of random variables X, Y and $Z: F_X, F_Y$ and F_Z the following properties are true:

- (i) if $F_X \leq F_Y$ and $F_Y \leq F_Z$, then $F_X \leq F_Z$ (transitivity);
- (ii) $F_X \leq F_X$ (reflexivity);
- (iii) if $F_X \leq F_Y$ and $F_Y \leq F_X$, then $F_X \equiv F_Y$ (antisymmetry).

If, moreover, for any given pair F_X and F_Y elements Ψ either $F_X \leq F_Y$ or $F_Y \leq F_X$, then such an order is considered a complete order.

Although the order relations used are often defined on sets of distribution functions, it is often convenient not to distinguish between the order relation for distribution functions and the corresponding relation for random variables.

Expanding the notation, we write $X \leq Y$, but in fact we mean $F_X \leq F_Y$. In other words, when we say that the risk X is less than the risk Y with respect to the stochastic order, we are actually arguing that this order is true for the corresponding distribution functions of these risks. Therefore, the joint distribution of X and Y does not matter here. Only their one-dimensional distributions are important. If $X \leq Y$, then $X \leq \tilde{Y}$ it is also true for any random variable with the same one-dimensional distribution as Y .

In such situations, we should not assume that the random variables being compared are defined on the same probability space. However, the question then remains: can we construct versions of random variables of interest that satisfy some additional properties on the same probability space?

Stochastic Dominance and VaR

To compare a pair of risks X and Y , it seems natural to resort to the concept VaR and consider X less dangerous than Y if

$VaR[X, p_0] \leq VaR[Y, p_0]$, for a given level of confidence probability p_0 . However, sometimes it is difficult to choose such p_0 that exactly matches your concerns about these risks, and it is possible that both $VaR[X, p_0] \leq VaR[Y, p_0]$ and $VaR[X, p_1] \geq VaR[Y, p_1]$ at the same time for two levels of confidence probabilities p_0 and p_1 . Therefore, it seems reasonable to adopt the following criterion: we place X before Y (it is less dangerous) if VaR for X is less than the corresponding VaR for Y , for any confidence probability level p .

Definition 2

Let X and Y be two random variables. Then they say that X is no more than Y in the sense of stochastic dominance, and denote this as $X \leq_{ST} Y$, if the inequality $VaR[X, p] \leq VaR[Y, p]$ is satisfied for all $p \in [0, 1]$.

Let's start with a few simple but important points.

Remark 1

A comparison $X \leq_{ST} Y$ is true only if the inequality $F_X(x) \geq F_Y(x)$ holds for all $x \in R$.

The proof of this fact obviously follows from the definition of the risk measure VaR .

Remark 2

If $Pr[X \leq Y] = 1$, then $X \leq_{ST} Y$.

(Hereafter, $Pr[A]$ denotes the probability of event A).

Proof

It follows from $Pr[X \leq Y] = 1$ that $F_X(x) \geq F_Y(x)$ for any x . Therefore, inequality $VaR[X, p] \leq VaR[Y, p]$ is satisfied for all $p \in [0, 1]$, which means $X \leq_{ST} Y$.

Remark 3

Obviously, the comparison $X \leq_{ST} Y$ is true only if $-Y \leq_{ST} -X$.

The following statement, proved by Kaas et al. [10], is also true.

Statement 1

Two random variables X and Y satisfy the relation $X \leq_{ST} Y$ if and only if there exist

random variables $\tilde{X}$ and $\tilde{Y}$ are such that $X = {}_d\tilde{X}$ and $Y = {}_d\tilde{Y}$ and with $Pr[X \leq Y] = 1$. (For the proof, see Kaas et al. [10]).

Here, the designation $X = {}_d\tilde{X}$ means that X and $\tilde{X}$ coincide in distribution.

Stochastic dominance and related properties are used in modern insurance to determine premium levels for insurance of various risks. In risk management, it is used in determining the order (priority) of risk management, as well as in determining the funds allocated to manage a particular risk.

In reinsurance of a set of risks, these methods are used to determine the reinsurance level at which reinsurance of each of the risks will be implemented.

The standard references on stochastic order $\leq_{ST}$, including appendices, are books by Lehmann [11], Marshall, Olkin [12], Ross [13] and Stoyan [14].

Examples of Stochastic Dominance for Risks with Different Distribution Functions

Example 1. Risks with an even distribution of losses.

Let us denote the set of random variables uniformly distributed on the segment $[a, b]$ through $Uni[a, b]$. And let's consider two risks X and Y , moreover, $X \in Uni[a, b]$ and $Y \in Uni[c, d]$.

In this case, it is possible to prove (see Appendix 1) the validity of the following sentence.

Proposal 1

Suppose we have two potential risks X and Y , additionally $X \in Uni[a, b]$, and $Y \in Uni[c, d]$.

The relationship $X \leq_{ST} Y$ is true if and only if one of the following situations is met:

1) $b \leq c$, that is, if the right end of the segment is $[a, b]$ is to the left of the left end of the segment $[a, b]$;

2) $b \geq c$, $b - a \geq d - c$ and $d \geq b$, that is, if the length of the segment $[a, b]$ is greater than the length of the segment $[c, d]$, the left end of the segment $[c, d]$ it is located to the right of the right end of the segment $[a, b]$, and the right end of the segment $[c, d]$ it is located to the right of the right end of the segment $[a, b]$.

3) $b \geq c \geq a$, $b - a \leq d - c$, that is, if the length of the segment is $[a, b]$ less than the length of the segment $[c, d]$, and the left end of the segment $[c, d]$ is located inside the segment $[a, b]$.

Example 2. Risks with a shifted exponential loss distribution.

Let's denote the set of random variables with the following distribution function:

$$F(x) = 1 - \exp(-k(x-a)), \text{ if } x \geq a \\ F(x) = 0, \text{ in other cases,}$$

where $k > 0$, and $a \in \mathbb{R}$ through $Exp(k, a)$ (in particular, when $a = 0$, the usual exponential distribution with a parameter $k > 0$ is obtained). And consider two risks X and Y , moreover $X \in Exp(k_1, a_1)$, and $Y \in Exp(k_2, a_2)$.

First, we obtain the expression for $Var[X, p]$.

According to the definition $Var[X, p]$, $p = Pr[X < Var[X, p]] = 1 - \exp(-k(Var[X, p] - a))$, means, $1 - p = \exp(-k(Var[X, p] - a))$, which means that

$$Var[X, p] = a - \frac{\ln(1-p)}{k}. \quad (1)$$

The ratio $X \leq_{ST} Y$ is equivalent to the inequality $Var[X, p] \leq Var[Y, p]$ for all p from the interval $[0, 1]$, which, according to formula (1), is equivalent to fulfilling the inequality

$$a_1 - \frac{\ln(1-p)}{k_1} \leq a_2 - \frac{\ln(1-p)}{k_2} \quad (2)$$

For any p from $[0, 1]$.

Let us choose, in particular, $p = 0$, then from inequality (2) we obtain $a_1 \leq a_2$.

However, then inequality (2) equals the inequality

$$\frac{1}{k_1} - \frac{1}{k_2} \leq \frac{a_1 - a_2}{\ln(1-p)}.$$

But it's easy to understand that the right-hand side of this inequality is a function of the variable p , which takes only positive values, filling in the interval $(0, +\infty)$.

Hence, in order to fulfill inequality (2) for all p from the interval $[0, 1]$, it is necessary and sufficient that the inequality be fulfilled

$$\frac{1}{k_1} - \frac{1}{k_2} \leq 0 \text{ or } k_2 \leq k_1.$$

Thus, the following proposal is true.

Proposal 2

Let's consider we have two risks X and Y , moreover $X \in \text{Exp}(k_1, a_1)$, and $Y \in \text{Exp}(k_2, a_2)$.

Then the ratio $X \leq_{ST} Y$ is satisfied only if the following conditions $a_1 \leq a_2$ and $k_2 \leq k_1$ are met.

Example 3. Risks with a normal loss distribution.

Let's denote the set of random variables with a normal loss distribution with parameters m and s by $\text{Nor}(m, s)$. And consider two risks X and Y , moreover $X \in \text{Nor}(m_1, s_1)$, and $Y \in \text{Nor}(m_2, s_2)$.

As is known (see, for example, [14]), if $X \in \text{Nor}(m, s)$, then

$$\text{VaR}[X, p] = m + k_p s,$$

where k_p — quantile of a standard normal distribution with confidence probability p .

The ratio $X \leq_{ST} Y$ is equivalent to the inequality $\text{VaR}[X, p] \leq \text{VaR}[Y, p]$

for all p from the interval $[0, 1]$. Which is equivalent to fulfilling the inequality $m_1 + k_p s_1 \leq m_2 + k_p s_2$ for any p in the range $[0, 1]$.

It follows that

$$k_p [s_1 - s_2] \leq m_2 - m_1. \quad (3)$$

Consider the first case: 1) $s_1 > s_2$. Then inequality (3) is equivalent to fulfilling the inequality

$$k_p \leq \frac{m_2 - m_1}{s_1 - s_2}$$

for any p in the range $[0, 1]$, which is impossible based on the properties of the quantile of the standard normal distribution. Consider the second case: 2) $s_1 < s_2$, then inequality (3) is equivalent to fulfilling the inequality

$$k_p \geq \frac{m_2 - m_1}{s_1 - s_2}$$

for any p in the range $[0, 1]$, which is also impossible based on the properties of the quantile of the standard normal distribution.

Consider the third case: 3) $s_1 = s_2$. Then from inequality (3) we obtain $m_2 \geq m_1$.

Thus, the following proposal 3 is true.

Proposal 3

Let's consider that we have two risks X and Y , moreover

$X \in \text{Nor}(m_1, s_1)$, and $Y \in \text{Nor}(m_2, s_2)$.

Then:

If $s_1 \neq s_2$, then risks X and Y stochastically not comparable;

If $s_1 = s_2$, then $X \leq_{ST} Y$ if and only if $m_2 \geq m_1$.

That is, normally distributed risks are comparable from the point of view of stochastic comparison, only if the volatility of these risks is equal, and in this case one of the risks does not exceed the other, if the expected loss on it does not exceed the expected loss on the second.

Example 4. Risks with a triangular distribution of losses.

Let's denote the set of random variables with a triangular distribution on the segment $[a, b]$ and a mode m through $\text{Tr}[a, b, m]$. And consider the two risks X and Y , where $X \in \text{Tr}[a, b, m_1]$ and $Y \in \text{Tr}[a, b, m_2]$, meanwhile $m_1 < m_2$.

In this case, it is possible to prove (see Appendix 2) the validity of the following sentence.

Proposal 4

Let us have two risks X and Y , where $X \in \text{Tr}[a, b, m_1]$, and $Y \in \text{Tr}[a, b, m_2]$. Meanwhile $m_1 < m_2$.

Then the relation $X \leq_{ST} Y$ holds in two cases:

$$\begin{aligned} &\text{if } m_1 < m_2 \leq a(1-p) + pb; \\ &\text{if } a(1-p) + pb \leq m_1 < m_2. \end{aligned}$$

In the third case, when

$$m_1 \leq a(1-p) + pb \leq m_2,$$

relation $X \leq_{ST} Y$ is not true, and dominance $Y \leq_{ST} X$ can be either performed or not — depending on the ration of parameters a, b, m_1, m_2 .

It is interesting how the stochastic dominance ratio can be expressed in terms of risk “ VaR to the power of t ”, $VaR^{(t)}[X, p]$ introduced and investigated in [8]. For any true value $t \geq 1$. The answer to this question is given in the following statement:

Statement 2

If $X \leq_{ST} Y$, then inequality

$VaR^{(t)}[X, p] \leq VaR^{(t)}[Y, p]$ is true under all $t \geq 1$ for all $p \in [0, 1]$.

If inequality $VaR^{(t)}[X, p] \leq VaR^{(t)}[Y, p]$ is true under at least one value $t = t_0 \geq 1$ for all $p \in [0, 1]$, then $X \leq_{ST} Y$.

The proof of the claim can be found in *Appendix 3*.

It follows from statement 2 that risk $VaR^{(t)}$ are applicable to determine the stochastic dominance of risks. And the risk X is less risky than Y in the sense of stochastic dominance if the inequality $VaR^{(t)}[X, p] \leq VaR^{(t)}[Y, p]$ is true for at least one value $t = t_0 \geq 1$ for all $p \in [0, 1]$.

Based on this, it is natural to introduce a definition:

Definition 3

Let X and Y be two random variables representing the corresponding risk factors. Then it is said X to be less than Y in the sense of stochastic dominance of degree t , and denoted as $X \leq_{ST^{(t)}} Y$, if the inequality $VaR^{(t)}[X, p] \leq VaR^{(t)}[Y, p]$ is satisfied for all $p \in [0, 1]$.

Then Statement 1 can be reformulated as follows:

Statement 3

If $X \leq_{ST} Y$, then comparison $X \leq_{ST^{(t)}} Y$ is true for all $t \geq 1$.

If comparison $X \leq_{ST^{(t)}} Y$ is true for at least one value $t = t_0 \geq 1$, then $X \leq_{ST} Y$.

Stochastic dominance can also be characterized by relative inverse distribution functions defined as $x \rightarrow VaR[X, F_Y(x)]$.

This is nothing more than VaR from X at the level of confidence probability $p = F_Y(x)$.

In [14], the following Statement (propositions 3.3.2 and 3.3.3) was proved,

representing necessary and sufficient conditions for stochastic dominance.

Statement 4

For two risks X и Y , the ratio $X \leq_{ST} Y$ is true if and only if $VaR[X, F_Y(x)] \leq x$ for all x .

Let X and Y risks with continuous distribution functions. In this case, the ratio is $X \leq_{ST} Y$ true if and only if $F_X(VaR[Y, p]) \geq p$ for all $0 < p < 1$.

Combining statements 4 and 3, we can conclude that the following Statement is true.

Statement 5

Let X and Y — be random variables with continuous distribution functions. If at least one of the conditions is met:

$VaR[X, F_Y(x)] \leq x$ for all x ;

$F_X(VaR[Y, p]) \geq p$ for all $0 < p < 1$,

Then comparison $X \leq_{ST^{(t)}} Y$ is true under all $t \geq 1$.

If comparison $X \leq_{ST^{(t)}} Y$ is true for at least one value $t = t_0 \geq 1$, then the following two conditions will be fulfilled simultaneously:

$VaR[X, F_Y(x)] \leq x$ for all x ;

$F_X(VaR[Y, p]) \geq p$ for all $0 < p < 1$.

In [14] the following Statement is presented:

Statement 6

For any random variables X and Y , the comparison $X \leq_{ST} Y$ is true if and only if the inequality $r_g[X] \leq r_g[Y]$ holds for all measures of distortion risk r_g with a nonzero distortion function g .

Then the validity of the following statement follows from statement 3:

Statement 7

For any random variables X and Y the comparison $X \leq_{ST^{(t)}} Y$ is true if and only if the inequality $r_g[X] \leq r_g[Y]$ holds for all measures of distortion risk r_g with a non-zero distortion function g .

That is, if a comparison $X \leq_{ST^{(t)}} Y$ is true for some t , then those making decisions about comparing risks based on $X \leq_{ST^{(t)}} Y$ will prefer risk X over risk Y , and, for any measure, risk

of distortion r_g with $g \neq 0$ and vice versa. The following Statement is given in [14] (Statement 3.3.4):

Statement 8

For any random variables X and Y , a comparison $X \leq_{ST} Y$ is true if and only if the inequality $E[v(X)] \leq E[v(Y)]$ holds for all non-decreasing functions v , for which these expected values exist.

Then the validity of the following statement follows from statement 3:

Statement 9

For any random variables X and Y , a comparison $X \leq_{ST^{(t)}} Y$ is true if and only if the inequality $E[v(X)] \leq E[v(Y)]$ holds for all non-decreasing functions v for which these expected values exist.

**PREMIUMS ON STOP-LOSS
CONTRACTS FOR DIFFERENT RISKS
AND THE STOCHASTIC ORDER
BETWEEN THEM**

In the financial and insurance industries, so-called stop-loss contracts are often used, which allow you to receive compensation when a certain risk value X exceeds a certain threshold value t . The value (where $Y_+ = \max\{Y, 0\}$) represents the size of the excess itself over the threshold t .

Definition 4

The function $p_X(t) = E[(X - t)_+]$ is called stop loss conversion X .

Premiums for stop-loss contracts are calculated based on the value of the stop-loss conversion. For more information on the use and calculation of stop-loss premiums, see Kaas [16].

It is useful to provide some properties of the “stop loss” p_X conversion (see [14] property 1.7.2).

Property

Let's consider that $E[|X|] < +\infty$. The stop-loss conversion X has the following properties:

(i) it is decreasing and convex;

(ii) $\lim_{t \rightarrow \infty} p_X(t) = 0$ and $\lim_{t \rightarrow -\infty} \{p_X(t) + t\} = E[X]$.

The next result regarding the order ratio $\leq_{ST}$ is related to the stop loss conversion.

Statement 10 (See [14])

Given two random variables X and Y , $X \leq_{ST} Y$ if and only if $p_Y - p_X$ does not increase in the area $(-\infty, +\infty)$.

Statement 2 implies the validity of the following statement:

Statement 11

For given two random variables X and Y $X \leq_{ST^{(t)}} Y$ for a certain t if and only if $p_Y - p_X$ does not increase in the range $(-\infty, +\infty)$.

It is interesting to interpret the latest statements from the perspective of risk management.

They point out that when $X \leq_{ST^{(t)}} Y$ is true, the difference between their respective stop loss premiums decreases as the threshold level t increases. Since $\lim_{t \rightarrow +\infty} (E[(Y - t)_+] - E[(X - t)_+]) = 0$, the largest difference between premiums occurs when $t = 0$ (and is equal to $E[Y] - E[X]$). And this difference decreases as the threshold level increases. Thus, evaluating stop-loss contracts for risks that are comparable in accordance with the order $\leq_{ST^{(t)}}$ leads to very different premiums at small thresholds, but this difference will decrease as this level increases.

**Comparing deficits with stochastic dominance:
The order of variance**

Now we will take a different path, and we will compare deficits instead of risks X and Y . This leads to the following definition.

Definition 5

For these two risks X and Y , it is said that X is lower than Y in the order of variance, and denoted as $X \leq_{DISP} Y$

if $(X - VaR[X, p])_+ \leq_{ST} (Y - VaR[Y, p])_+$ for all $0 < p < 1$.

Early references to the relation of order $\leq_{DISP}$ are related to the works of Doksum [17], Deshpande, Kochar [18]. See also Bartoszewicz [19, 20], Munoz-Perez [21] and Landsberger, Meilijson [22, 23].

Remark

The relation $\leq_{\text{DISP}}$ is reflexive and transitive, but not antisymmetric on the set of distribution functions. Indeed, if $X \leq_{\text{DISP}} Y$ and $Y \leq_{\text{DISP}} X$ are both true at the same time, it means that $Y = {}_d X + c$ for some constant c . This is easily seen from the fact that

$$(X' - \text{VaR}[X', p]) = {}_d (X - \text{VaR}[X, p])$$

if $X' = {}_d X + c$.

Using the concept of “ VaR to the power of t ”, we introduce a more general concept of the order of variance of degree t for any $t \geq 1$.

Definition 6

For these two risks X and Y , we will say that X is less than Y in the order of variance of degree t , $t \geq 1$, and denote as $X \leq_{\text{DISP}^{(t)}} Y$, if and only if $(X - \text{VaR}^{(t)}[X, p])_+ \leq_{ST} (Y - \text{VaR}^{(t)}[Y, p])_+$ for all $0 < p < 1$.

It is interesting how the ratio of the dispersed order can be expressed in terms of the ratio of the dispersed order of degree t for any real value $t \geq 1$.

Statement 12

If $X \prec_{\text{DISP}} Y$, then ratio $X \leq_{\text{DISP}^{(t)}} Y$ is true for all $t \geq 1$. If the ratio $X \leq_{\text{DISP}^{(t)}} Y$ is true for at least one $t = t_0 \geq 1$ value, then $X \leq_{\text{DISP}} Y$.

The proof is given in *Appendix 4*.

It follows from *Statement 1* that risk measures $\text{VaR}^{(t)}$ are applicable to determine the stochastic dominance of risks. And the risk X is less risky than Y in the sense of stochastic dominance if the inequality $\text{VaR}^{(t)}[X, p] \leq \text{VaR}^{(t)}[Y, p]$ is true for at least one $t = t_0 \geq 1$ value for all $p \in [0, 1]$.

ORDER OF VARIANCE AND SPREADS VaR

Some stochastic orders are based on the distances between the corresponding VaR basic random variables. The dispersion order has this property related to the difference between VaR (see [14], Property 3.3.60).

Property

For given two random variables X and Y , $X \leq_{\text{DISP}} Y$ is true if and only if

$\text{VaR}[X, p_2] - \text{VaR}[X, p_1] \leq \text{VaR}[Y, p_2] - \text{VaR}[Y, p_1]$ each time $0 < p_1 \leq p_2 < 1$, or equivalently if $\text{VaR}[Y, p] - \text{VaR}[X, p]$ increases by $p \in (0, 1)$.

It can be seen from this property that $\leq_{\text{DISP}}$ is indeed the order of variance.

Risk measures $\text{VaR}^{(t)}$ have a similar property.

Statement 13

For given two random variables X and Y , $X \leq_{\text{DISP}} Y$ is true if and only if

$$\begin{aligned} \text{VaR}^{(t)}[X, p_2] - \text{VaR}^{(t)}[X, p_1] &\leq \\ &\leq \text{VaR}^{(t)}[Y, p_2] - \text{VaR}^{(t)}[Y, p_1] \end{aligned}$$

for any $t \geq 1$ every time when $0 < p_1 \leq p_2 < 1$ or, equivalently, if and only if

$\text{VaR}^{(t)}[Y, p] - \text{VaR}^{(t)}[X, p]$ increases by $p \in (0, 1)$.

The proof is given in *Appendix 5*.

The following useful Statement follows from the previous Property:

Statement 14

For the given two random variables X and Y , which satisfy the relation $X \leq_{\text{DISP}^{(t)}} Y$ is true for a certain $t \geq 1$ value, the relation is also true $X \leq_{\text{ST}} Y$ if the value of the left boundary point of the value carrier Y is greater than or equal to the value of the left boundary point of the value carrier X .

Proof

If $X \leq_{\text{DISP}^{(t)}} Y$, then it follows from the previous statement that

$$\begin{aligned} \text{VaR}^{(t)}[Y, p] - \text{VaR}^{(t)}[X, p] &\geq \\ &\geq \text{VaR}^{(t)}[Y, 0] - \text{VaR}^{(t)}[X, 0] \geq 0. \end{aligned}$$

Moreover, the latter inequality obviously follows from the fact that the value of the left boundary point of the magnitude carrier Y is greater than or equal to the value of the left boundary point of the magnitude carrier X and the meaning of the risk measure $\text{VaR}^{(t)}$.

Examples of Variance Dominance for Risks with Different Distribution Functions

Example 5. Risks with an even distribution of losses. Let's consider two risks X and Y , moreover $X \in \text{Uni}[a, b]$, and $Y \in \text{Uni}[c, d]$.

But for risks with distributions from $Uni[a, b]$, the formula $VaR[X, p] = a + (b - a)p$ is true.

Therefore,

$$VaR[Y, p] - VaR[X, p] = c + (d - c)p - a - (b - a)p = c - a + (a + d - (b + c))p.$$

It can be seen that this is a function of the variable p is non-decreasing if and only if $a + d \geq b + c$, i.e. $d - c \geq b - a$.

Thus, the following Proposal is valid:

Proposal 5

Let's consider we have two risks X and Y , moreover $X \in Uni[a, b]$, and $Y \in Uni[c, d]$.

In this case, the ratio $X \leq_{DISP} Y$ is fulfilled if and only if $d - c \geq b - a$, i.e. the length of the segment $[c, d]$ is not less than the length of the $[a, b]$ segment.

Example 6. Risks with a shifted exponential loss distribution.

Let's consider two risks X and Y , moreover $X \in Exp(k_1, a_1)$, and $Y \in Exp(k_2, a_2)$.

As we know,

$$\text{for } X \in Exp(k, a) \quad VaR[X, p] = a - \frac{\ln(1-p)}{k}.$$

Therefore,

$$VaR[Y, p] - VaR[X, p] = a_2 - a_1 + \ln(1-p) \left(\frac{1}{k_1} - \frac{1}{k_2} \right).$$

It can be seen from this that this function of the variable p is non-decreasing if and only if $k_2 \leq k_1$.

Thus, the following Proposal is true:

Proposal 6

Let's consider we have two risks X and Y , moreover $X \in Nor(m_1, s_1)$, and $Y \in Exp(k_2, a_2)$.

Then the relation $X \leq_{DISP} Y$ holds if and only if $k_2 \leq k_1$.

Example 3. Risks with a normal distribution of losses.

Let's consider two risks X and Y , moreover $X \in Nor(m_1, s_1)$, and $Y \in Nor(m_2, s_2)$.

As we know, (see Example [14]) if $X \in Nor(m, s)$, then $VaR[X, p] = m + k_p s$.

Therefore,

$$VaR[Y, p] - VaR[X, p] = m_2 - m_1 + k_p(s_2 - s_1).$$

It can be seen from this that, since the quantile k_p is an increasing function of p , this

function of the variable p is non-decreasing if and only if $s_2 \geq s_1$.

Thus, the following Proposal is true:

Proposal 7

Let's consider we have two risks X and Y , moreover $X \in Nor(m_1, s_1)$, and $Y \in Nor(m_2, s_2)$.

Then the relation $X \leq_{DISP} Y$ holds if and only if $s_2 \geq s_1$.

Example 4. Risks with a triangular distribution of losses.

Let's consider two risks X and Y , moreover $X \in Tr[a, b, m_1]$, and $Y \in Tr[a, b, m_2]$.

It can be proved (see Appendix 6) that the following Proposal is true:

Proposal 8

Let's consider we have two risks X and Y , moreover $X \in Tr[a, b, m_1]$ and $Y \in Tr[a, b, m_2]$.

Then the relation $X \leq_{DISP} Y$ holds only if $m_1 < m_2$.

THE HAZARD RATE AND THE ORDER OF DISPERSION

When the possibility of a large loss is assessed, the importance of the value of the dual distribution function (survival function) $\bar{F}_X(x)$ is obvious, which estimates the probability of a loss of X exceeding the value of x . Many studies have noted the importance of a value called the hazard rate (see, for example, [24]), which can help in estimating the weight of the right tail of the loss distribution.

Definition 7. With a non-negative value of random losses X , with a distribution density $f_X(x)$, the value of the hazard rate is defined as a function of

$$r_X(x) = \frac{f_X(x)}{\bar{F}_X(x)}, \quad x \geq 0.$$

In reliability theory, the hazard rate is called the failure rate, and in life insurance, the mortality rate.

From the definition, it is easy to get such an idea for the danger rate:

$$r_X(x) = \lim_{\Delta x \rightarrow 0} \Pr[x < X \leq x + \Delta x | X > x] / \Delta x,$$

where $Pr[A|B]$ is used to denote the conditional probability of event A , provided that event B is executed.

From the latter ratio, the following interpretation of the danger rate follows: it determines the probability of danger in x ("failures") under the condition of a favorable combination of circumstances ("survival") to x . Intuitively speaking, if r_x turns out to be small, then the distribution becomes with a heavy tail, if it turns out to be large, then the loss distribution becomes a distribution with a light tail.

From the latter ratio, the following two ratios are obtained, linking the values

$$r_x(x) = -\frac{1}{\bar{F}_x(x)} \frac{d}{dx} \bar{F}_x(x),$$

$$\bar{F}_x(x) = \exp\left(-\int_0^x r_x(t) dt\right), x \geq 0, \quad (4)$$

from which it follows that the hazard rate uniquely characterizes the distribution of danger. And vice versa: knowing the law of loss distribution, it is possible to calculate the hazard rate, for example, if the magnitude of the hazard X obeys a Pareto distribution with parameters $a > 0, t > 0, X \in Par(a, t)$, then it turns out that the hazard rate is equal (see, for example, [14])

$$r_x(x) = \frac{a}{t+x}. \quad (5)$$

We see that r_x is strictly decreasing from $r_x(0) = \frac{a}{t}$ to $r_x(+\infty) = 0$.

If $r_x(x) = k = const$, then from formulas (1) it turns out that

$$F_x(x) = e^{-kx}, \text{ if } x \geq 0,$$

$$F_x(x) = 0, \text{ if } x < 0,$$

that is, the random variable obeys an exponential distribution with a parameter called intensity $X \in Exp(k)$. That is, a special case of the constancy of the hazard rate is realized in the case of exponentially distributed random loss values and coincides with the intensity of such values.

The dispersion order has this property related to the difference between VaR (see [14], property 3.3.62).

Property

For given two random variables X and Y , $X \leq_{DISP} Y$ is valid if and only if

$$r_Y(VaR[Y, p]) \leq r_X(VaR[X, p]) \text{ for all } 0 < p < 1.$$

Example 7. Risks with an exponential distribution.

Let's consider two risks X and Y , moreover $X \in Exp(k_1)$, and $Y \in Exp(k_2)$.

Then it follows from the previous property that $X \leq_{DISP} Y$ is true if and only if $k_2 \leq k_1$.

Example 8. Risks with Pareto loss allocation.

Let's consider two risks X and Y , moreover $X \in Par(a_1, t_1)$, and $Y \in Par(a_2, t_2)$.

Then it follows from formula (5) and the previous property that $X \leq_{DISP} Y$ is true if and only if the inequality

$$\frac{a_2}{t_2 + VaR[Y, p]} \leq \frac{a_1}{t_1 + VaR[X, p]} \quad (6)$$

holds under all $0 < p < 1$.

We obtain an expression for the risk measure VaR for any random variable $X \in Par(a, t)$.

According to the definition $VaR[X, p]$, we have $\bar{F}_x(VaR[X, p]) = 1 - p$, i.e.

$$\left(1 + \frac{VaR[X, p]}{t}\right)^{-a} = 1 - p, \text{ which means that}$$

$$VaR[X, p] = t((1 - p)^{1/a} - 1).$$

Using this result, we can rewrite inequality (6) as

$$\frac{a_2}{t_2(1 - p)^{-1/a_2}} \leq \frac{a_1}{t_1(1 - p)^{-1/a_1}}. \quad (7)$$

Let's consider the function

$$g(a, t) = \frac{a}{t(1 - p)^{-1/a}} = \frac{a}{t} (1 - p)^{1/a}.$$

It is obvious that for the variable t it is a non-increasing function, and for the variable a it is a non-decreasing function, from which follows the next Proposal.

Proposal 9

Let's consider two risks X and Y , moreover $X \in Par(a_1, t_1)$, and $Y \in Par(a_2, t_2)$.

Then the ratio $X \leq_{DISP} Y$ holds if and only if $a_1 \geq a_2$ and $t_1 \leq t_2$.

It turns out that the generalization of the latter property to general relations $\leq_{DISP^{(t)}}$ is also true at any $t \geq 1$.

Statement 15

For given two random variables X and Y , $X \leq_{DISP^{(t)}} Y$ is true if and only if

$$r_Y(VaR^{(t)}[Y, p]) \leq r_X(VaR^{(t)}[X, p]).$$

For all $0 < p < 1$. (The proof is given in Appendix 7).

CONCLUSIONS

The intuitive concept of stochastic order between random variables has been developed in probability theory since the middle of the XXth century. This has led to the introduction of several related concepts of stochastic order, which have been studied in scientific literature. However, it was not until the 1990s that these concepts began to be applied in risk management and actuarial fields, making it possible to compare risk factors and the level of

associated hazards. This allowed practitioners and researchers in these fields to not only obtain estimates of potential losses, but also to compare risk factors themselves and determine the level of risk associated with them. In addition, it has been found that various types of stochastic orders, which can be applied to a variety of risks we are interested in, can be described using risk measures such as Value at Risk (VaR).

This paper is devoted to the description of some types of stochastic order between risks VaR and the study of their relationships with risk measures and measures of catastrophic risks VaR to the degree of t , $VaR^{(t)}$. In addition, for well-known examples of loss distributions that find wide applications in risk management and actuarial science, necessary and sufficient conditions have been obtained to compare risks subject to these distributions.

The results and methods of this work may be of interest to both practitioners and scientists involved in risk management and the actuarial field.

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APPENDICES

Appendix 1

Obviously, if the inequality $b \leq c$ is fulfilled, i.e. the distribution carrier $Uni[a, b]$ is entirely located below the distribution carrier $Uni[c, d]$, then the inequality $VaR[X, p] \leq VaR[Y, p]$ will be met for all $p \in [0, 1]$, which means $X \leq_{ST} Y$. Now let's assume that $b > c$. However, the ratio $X \leq_{ST} Y$ is equivalent to the inequality $VaR[X, p] \leq VaR[Y, p]$ for all p from the interval $[0, 1]$.

However, as is well known (for example, [15] Appendix 2), the formula $VaR[X, p] = a + (b - a)p$ is true for risks with distributions from $Uni[a, b]$. So, we have $a + (b - a)p \leq c + (d - c)p$, i. e. $p[b - a - (d - c)] \leq c - a$.

Let's consider two possible cases:

1) assume that $b - a \geq d - c$, i.e., the length of the segment $[a, b]$ is greater than the length of the segment $[c, d]$.

Then we get that the inequality $p \leq \frac{c - a}{(b - a) - (d - c)}$ must hold for any $p \in [0, 1]$, and this can only be true

if the inequality $\frac{c - a}{(b - a) - (d - c)} \geq 1$ is fulfilled, which is equivalent to $d \geq b$;

2) assume that $b - a < d - c$, i.e., the length of the segment $[a, b]$ is less than the length of the segment $[c, d]$.

Then we get that the inequality $p \geq \frac{c - a}{(b - a) - (d - c)}$ must hold for any $p \in [0, 1]$, and this can only be true

if the inequality $\frac{c - a}{(b - a) - (d - c)} \leq 0$ is fulfilled, which is equivalent to $c \geq a$.

Thus, Proposal 1 is true.

Appendix 2

The ratio $X \leq_{ST} Y$ is equivalent to the inequality $VaR[X, p] \leq VaR[Y, p]$ for all p from the interval $[0, 1]$. But, as is well known (for example, [15] Appendix 3), in the case of risks with distributions from $Tr[a, b, m]$ the formula $VaR[X, p] = b - \text{sqr}[(1 - p)(b - a)(b - m)]$, if $m \leq a(1 - p) + pb$ and $VaR[X, p] = a + \text{sqr}[p(b - a)(v - a)]$, is true if $m > a(1 - p) + pb$. (Hereafter $\text{sqr}[X]$ is the notation for the square root of X .) $m_1 < m_2 \leq a(1 - p) + pb$.

In this case, the inequality $VaR[X, p] \leq VaR[Y, p]$ is equivalent to the inequality

$$b - \sqrt{(1 - p)(b - a)(b - m_1)} \leq b - \sqrt{(1 - p)(b - a)(b - m_2)},$$

which, in turn, is equivalent to equitable inequality $m_1 \leq m_2$. $a(1 - p) + pb \leq m_1 < m_2$.

In this case, the inequality $VaR[X, p] \leq VaR[Y, p]$ is equivalent to the inequality

$$a + \sqrt{p(b - a)(m_1 - a)} \leq a + \sqrt{p(b - a)(m_2 - a)},$$

which, in turn, is equivalent to equitable inequality $m_1 \leq m_2$. $m_1 \leq a(1 - p) + pb \leq m_2$. In this case, the inequality $VaR[X, p] \leq VaR[Y, p]$ is equivalent to the inequality

$$b - \sqrt{(1 - p)(b - a)(b - m_1)} \leq a + \sqrt{p(b - a)(m_2 - a)} \text{ or } \sqrt{p(b - a)(m_2 - a)} + \sqrt{(1 - p)(b - a)(b - m_1)} \geq b - a,$$

however, at $p = 0$ we get the inequality $\sqrt{(b - a)(b - m_1)} \geq b - a$, which is equivalent to the obviously incorrect inequality $m_1 \leq a$. Thus, the ratio $X \leq_{ST} Y$ is not executed. But then may be the ratio $Y \leq_{ST} X$? is executed? However, this ratio is equivalent to fulfilling the inequality

$$b - \sqrt{(1 - p)(b - a)(b - m_1)} \leq a + \sqrt{p(b - a)(m_2 - a)}$$

for all p from the range $[0, 1]$. Or inequalities $\sqrt{p(b - a)(m_2 - a)} + \sqrt{(1 - p)(b - a)(b - m_1)} \leq b - a$, which, using elementary transformations of squaring both parts and reducing similar terms, reduces to the inequality

$$2\sqrt{(m_2 - a)(b - m_1)}\sqrt{p(1 - p)} \leq p(b + a - m_1 - m_2) + m_1 - a.$$

However, a numerical study shows that depending on the ratio of the parameter values a, b, m_1, m_2 it can either be performed for all p from the interval $[0, 1]$, or not performed for certain values of p from this interval.

For example, when $a = 0, b = 100, m_1 = m_2 = 50$ this inequality amounts to inequality $2\sqrt{50 \cdot 50} \sqrt{p(1-p)} \leq 50$, which is equivalent to the inequality $(p - \frac{1}{2})^2 \geq 0$, which obviously holds for all values of p .

However, when $a = 0, b = 100, m_1 = 1, m_2 = 99$ this inequality amounts to inequality $2\sqrt{99 \cdot 99} \sqrt{p(1-p)} \leq 1$, which is equivalent to the inequality $2\sqrt{50 \cdot 50} \sqrt{p(1-p)} \leq 50$, which is equivalent to the inequality $p^2 - p + \frac{1}{4 \cdot 99^2} \geq 0$, which obviously does not hold on the set of p values from the interval

$$(p_1, p_2) \subset [0, 1], p_1 = \frac{1}{2}(1 - \sqrt{1 - \frac{1}{99^2}}), p_2 = \frac{1}{2}(1 + \sqrt{1 - \frac{1}{99^2}}).$$

Thus, the validity of proposition 4 is proved.

Appendix 3

An arbitrary real number $t \geq 1$ can be represented as $t = k + e$, where k is a natural number, and a is a real number, and $0 \leq e < 1$. Obviously, k is an integer part of the number t , and e is its fractional part.

Then, according to [8], the following representation $VaR^{(t)}[X, p] = VaR[X, 1 - (1-p)^k(1-ep)]$ is true: However, since, according to definition 1, the inequality $VaR[X, q] \leq VaR[Y, q]$ is fulfilled for all

$q \in [0, 1]$, it must also be fulfilled if $q = 1 - (1-p)^k(1-ep)$,

i.e. $VaR[X, 1 - (1-p)^k(1-ep)] \leq VaR[Y, 1 - (1-p)^k(1-ep)]$

for all $p \in [0, 1]$, so, and $VaR^{(t)}[X, p] \leq VaR^{(t)}[Y, p]$.

Statement (i) is proved.

Now let's move on to the proof of statement (ii). First, consider the case of a natural degree. That is, let's assume that for some natural number n_0 , the inequality $VaR^{(n_0)}[X, p] \leq VaR^{(n_0)}[Y, p]$ is true for all $p \in [0, 1]$.

However, using the formula for VaR to the power of t , we have:

$$VaR[X, 1 - (1-p)^{n_0}] \leq VaR[Y, 1 - (1-p)^{n_0}] \text{ for all } p \in [0, 1].$$

But for any $q \in [0, 1]$ there is such $p \in [0, 1]$, that $q = 1 - (1-p)^{n_0}$.

Namely, this is $p = 1 - (1-q)^{1/n_0}$. So then the inequality $VaR[X, q] \leq VaR[Y, q]$ is true at any $q \in [0, 1]$. It means that $X \leq_{ST} Y$.

For an arbitrary $t_0 \geq 1$, it can be represented as $t_0 = k + e$, where k is the integer part of the number t_0 , and e is its fractional part.

If, for some t_0 , the inequality $VaR^{(t_0)}[X, p] \leq VaR^{(t_0)}[Y, p]$ holds for all $p \in [0, 1]$ then, using the representation (2) proved in [8], we obtain that $VaR[X, 1 - (1-p)^k(1-ep)] \leq VaR[Y, 1 - (1-p)^k(1-ep)]$ for all $p \in [0, 1]$.

But for any $q \in [0, 1]$ there is a single number $p \in [0, 1]$ that $q = 1 - (1-p)^k(1-ep)$, i.e. $(1-p)^k(1-ep) = 1 - q$.

This follows from the fact that the function $f(p) = (1-p)^k(1-ep)$ for any $k \geq 1$ and $e \in [0, 1]$ is a continuous non-increasing function $f(0) = 1$ and $f(1) = 0$

That is, this function performs a one-to-one mapping of the segment $[0, 1]$ onto itself. Which, in turn, follows from the fact that $f'(p) = -k(1-p)^{k-1}(1-ep) - e(1-p)^k = -(1-p)^{k-1}[k(1-ep) + e(1-p)] \leq 0$.

So then the inequality $VaR[X, q] \leq VaR[Y, q]$ is executed at any $q \in [0, 1]$. Which means, $X \leq_{ST} Y$.

Appendix 4

For an arbitrary real number $t \geq 1$, it can be represented as $t = k + e$, where k is a natural number and e is a real number, and $0 \leq e < 1$. Obviously, k is an integer part of the number t , and e is its fractional part.

Then, according to [8], the representation is valid:

$$VaR^{(t)}[X, p] = VaR[X, 1 - (1-p)^k(1-ep)].$$

However, since according to the definition of 2, if $X \leq_{DISP} Y$, the ratio

$$(X - VaR[X, q])_+ \leq_{ST} (Y - VaR[Y, q])_+ \text{ for all } 0 < q < 1.$$

But then it must also be satisfied with a particular meaning $q = 1 - (1 - p)^k(1 - ep)$, i.e. $(X - VaR[X, 1 - (1 - p)^k(1 - ep)])_+ \leq_{ST} (Y - VaR[Y, 1 - (1 - p)^k(1 - ep)])_+$ for all $p \in [0, 1]$, and therefore $X \leq_{DISP^{(t)}} Y$.

Statement (i) is proved.

Now let's move on to the proof of statement (ii). First, consider the case of a natural degree. That is, let's assume that for some natural value n_0 , the ratio is true $X \leq_{DISP^{(n_0)}} Y$. It means that $(X - VaR^{(n_0)}[X, q])_+ \leq_{ST} (Y - VaR^{(n_0)}[Y, q])_+$

Then we get:

$$(X - VaR[X, 1 - (1 - q)^{n_0}])_+ \leq_{ST} (Y - VaR[Y, 1 - (1 - q)^{n_0}])_+ \text{ for all } q \in [0, 1].$$

But for any $p \in [0, 1]$ there is such $q \in [0, 1]$, that $p = 1 - (1 - q)^{n_0}$.

Namely, it is $q = 1 - (1 - p)^{1/n_0}$. In this way, then the ratio $(X - VaR[X, p])_+ \leq_{ST} (Y - VaR[Y, p])_+$ holds for any $p \in [0, 1]$. Which means, $X \leq_{DISP} Y$.

For an arbitrary real number $t_0 \geq 1$, it can be represented as $t_0 = k + e$, where k is the integer part of the number t_0 , and e is its fractional part.

If, for some t_0 , the relation is true for all $p \in [0, 1]$, then $(X - VaR^{(t_0)}[X, q])_+ \leq_{ST} (Y - VaR^{(t_0)}[Y, q])_+$.

Then we get:

$$(X - VaR[X, 1 - (1 - q)^k(1 - eq)])_+ \leq_{ST} (Y - VaR[Y, 1 - (1 - q)^k(1 - eq)])_+ \text{ for all } q \in [0, 1].$$

But there is a single number $q \in [0, 1]$ for any $p \in [0, 1]$, which is $p = 1 - (1 - q)^k(1 - eq)$, i.e. $(1 - q)^k(1 - eq) = 1 - p$.

This follows from the fact that the function $f(q) = (1 - q)^k(1 - eq)$ for any $k \geq 1$ and $e \in [0, 1]$ is a continuous non-increasing function, and $f(0) = 1$ and $f(1) = 0$. That is, this function performs a one-to-one mapping of the segment $[0, 1]$ onto itself. Which, in turn, follows from the fact that

$$f'(q) = -k(1 - q)^{k-1}(1 - eq) - e(1 - q)^k = -(1 - q)^{k-1}[k(1 - eq) + e(1 - q)] \leq 0.$$

Thus, then the ratio $(X - VaR[X, p])_+ \leq_{ST} (Y - VaR[Y, p])_+$ holds for any $p \in [0, 1]$. Which means $X \leq_{DISP} Y$.

Appendix 5

For an arbitrary $t \geq 1$, it can be represented as $t = k + e$, where k is the integer part of the number t , and e is its fractional part.

Consider a random variable $X_p^{(t)} = (X - VaR^{(t)}[X, p])_+$.

Then $VaR^{(t)}[X_p^{(t)}, q] = VaR[X_p^{(t)}, 1 - (1 - q)^k(1 - eq)] = 0$, if $q \leq p$ and $VaR^{(t)}[X_p^{(t)}, q] = VaR^{(t)}[X, q] - VaR^{(t)}[X, p]$, if $q > p$.

However $X \leq_{DISP} Y$, if and only if $X_p^{(1)} \leq_{ST} Y_p^{(1)}$, according to statement 2, is equivalent to the ratio $X_p^{(t)} \leq_{ST} Y_p^{(t)}$, i.e. $VaR^{(t)}[X_p^{(t)}, q] \leq VaR^{(t)}[Y_p^{(t)}, q]$ for arbitrary $t \geq 1$ and $0 < q < 1$. And this, in turn, is equivalent to $VaR^{(t)}[X, q] - VaR^{(t)}[X, p] \leq VaR^{(t)}[Y, q] - VaR^{(t)}[Y, p]$ under any $0 < p < q < 1$.

Appendix 6

The ratio $X \leq_{ST} Y$ is equivalent to the inequality $VaR[X, p] \leq VaR[Y, p]$ for all p from the interval $[0, 1]$. But, as is well known (for example, [15] Appendix 3), in the case of the formula $VaR[X, p] = b - \sqrt{p(1-p)(b-a)(b-m)}$, if $m \leq a(1-p) + pb$, and $VaR[X, p] = a + \sqrt{p(1-p)(b-a)(b-m)}$ if $m > a(1-p) + pb$ is true for risks with distributions from $Tr[a, b, m]$

1) Let's consider three cases: $m_1 < m_2 \leq a(1-p) + pb$ then

$$\begin{aligned} VaR[Y, p] - VaR[X, p] &= \sqrt{(1-p)(b-a)(b-m_1)} - \sqrt{(1-p)(b-a)(b-m_2)} = \\ &= \sqrt{(1-p)(b-a)}(\sqrt{b-m_1} - \sqrt{b-m_2}) \geq 0. \end{aligned}$$

From this it can be seen that this function of the variable p is non-decreasing and therefore $X \leq_{\text{DISP}} Y$;

2) $a(1-p) + pb \leq m_1 < m_2$. In this case

$$\begin{aligned} VaR[Y, p] - VaR[X, p] &= \sqrt{p(b-a)(m_2-a)} - \sqrt{p(b-a)(m_1-a)} = \\ &= \sqrt{p(b-a)}(\sqrt{m_2-a} - \sqrt{m_1-a}) \geq 0. \end{aligned}$$

From this it can be seen that this function of the variable p is non-decreasing and therefore $X \leq_{\text{DISP}} Y$;

3) $m_1 \leq a(1-p) + pb \leq m_2$.

In this case

$$\begin{aligned} VaR[Y, p] - VaR[X, p] &= a + \sqrt{p(b-a)(m_2-a)} - b + \sqrt{(1-p)(b-a)(b-m_1)} = \\ &= a - b + \sqrt{b-a}(\sqrt{p(m_2-a)} - \sqrt{(1-p)(b-m_1)}). \end{aligned}$$

Let's consider the function

$$f(p) = \sqrt{x_0 p} + \sqrt{(1-p)x_1}, \quad x_0 = m_2 - a, \quad x_1 = b - m_1.$$

Then

$$f'(p) = \frac{\sqrt{x_0}}{2\sqrt{p}} - \frac{\sqrt{x_1}}{2\sqrt{1-p}} = \frac{\sqrt{x_0(1-p)} - \sqrt{x_1 p}}{2\sqrt{p(1-p)}},$$

therefore, the inequality $f'(p) \geq 0$

Is equivalent to the inequality $\sqrt{x_0(1-p)} \geq \sqrt{x_1 p}$, which means that $p \leq \frac{x_0}{x_1 + x_0} = \frac{m_2 - a}{b - m_1 + m_2 - a}$.

However, in case (3), the inequalities $m_1 \leq a(1-p) + pb \leq m_2$ hold which are equivalent to the inequalities

$$\frac{m_1 - a}{b - a} \leq p \leq \frac{m_2 - a}{b - a} \text{ and because the fairness of inequality } \frac{m_2 - a}{b - m_1 + m_2 - a} \leq \frac{m_2 - a}{b - a} \text{ is obvious, then according}$$

to the property the ratio $X \leq_{\text{DISP}} Y$ hold.

Appendix 7

From statement 13 and the formula $VaR^{(t)}[X, p] = VaR[X, 1 - (1-p)^m(1-ep)]$, where the number t is represented as the sum of its integer and fractional parts $t = m + e, 0 \leq e < 1$ (see [8]), it follows that

$$\begin{aligned} 0 &\leq \frac{d}{dp}(VaR^{(t)}[Y, p] - VaR^{(t)}[X, p]) = \frac{m(1-p)^{m-1}(1-ep) + e(1-p)^m}{f_Y(VaR^{(t)}[Y, p])} - \\ &- \frac{m(1-p)^{m-1}(1-ep) + e(1-p)^m}{f_X(VaR^{(t)}[X, p])} = (1-p)^{m-1}(m(1-ep) + e(1-p)) \left(\frac{1}{f_Y(VaR^{(t)}[Y, p])} - \frac{1}{f_X(VaR^{(t)}[X, p])} \right). \end{aligned}$$

And since $(1-p)^{m-1}(m(1-ep) + e(1-p)) \geq 0$ at $0 < p < 1$, it follows that $f_Y(VaR^{(t)}[Y, p]) \leq f_X(VaR^{(t)}[X, p])$, which means that

$$\frac{f_Y(VaR^{(t)}[Y, p])}{(1-p)^m(1-ep)} \leq \frac{f_X(VaR^{(t)}[X, p])}{(1-p)^m(1-ep)}.$$

However, since, by definition, $VaR^{(t)}[X, p] = \bar{F}_X(VaR^{(t)}[X, p]) = (1-p)^m(1-ep)$, this means that

$$\frac{f_Y(VaR^{(t)}[Y, p])}{\bar{F}_Y(VaR^{(t)}[Y, p])} \leq \frac{f_X(VaR^{(t)}[X, p])}{\bar{F}_X(VaR^{(t)}[X, p])}.$$

And therefore, using the definition of the hazard rate, we obtain: $r_Y(VaR^{(t)}[Y, p]) \leq r_X(VaR^{(t)}[X, p])$.