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Fractal Asset Pricing Models for Financial Risk Management

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ABSTRACT

The article presents the analysis findings of the problems and prospects of using the fractal markets theory to mathematically predict the price dynamics of assets as part of a financial risk management strategy. The aim of the article is to find out the features of value of bank assets and to develop recommendations for assessing financial risks based on mathematical methods for forecasting economic processes. Theoretical and empirical research methods were used to achieve the aim. The article reveals the features of mathematical modeling of economic processes related to asset pricing in a volatile market. It was proved that using financial mathematics in banking contributes to the stable development of the economy. Mathematical modeling of the price dynamics of financial assets is based on a substantive hypothesis and supported by an adequate apparatus of fractal pair pricing models in order to reveal specific market relations of business entities. According to the authors, the prospects of using forecast models to minimize the financial risks of derivative financial instruments are positive. The authors concluded that the considered methods contribute to managing financial risks and improving forecasts, including operations with derivatives. Besides, the studied fractal volatility parameters proved the predictive power regarding extreme events in financial markets, such as the bankruptcy of Lehman Brothers investment bank in 2008. The relevance of the article is due to the fact that the favorable investment climate and the use of modern financing methods largely depend on the effective financial risk management.

Keywords: banking; asset valuation; economic and mathematical methods; financial risk management; hedging

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INTRODUCTION

As has been demonstrated globally, the traditional approach to the study of asset price dynamics is based on revealing economic trends and mathematical modeling of these trends to manage financial risks. For example, the classical Black-Scholes-Merton model is associated with the efficient market hypothesis (EMH) that assumes that the asset price is due to multiple random factors. The mathematical model of asset price dynamics contributes to disclosing its features. Using this model minimizes financial risks and ensures banking security in a volatile market.

Over the past century, financial mathematics has proved that to be viable, a mathematical model must be based on a substantive hypothesis and supported by an adequate mathematical apparatus. Models that do not have these two components, “unpaired” models, turn out to be unviable.

For example, the mathematical apparatus used by Bachelier in 1900 in his dynamic pricing model was ahead of his time, and the Bachelier model remained unclaimed for more than 60 years. After the efficient market hypothesis was developed, the Bachelier model became the basis for the modern pricing models.

In a sense, the opposite is exemplified by the fractal market hypothesis that appeared simultaneously with the efficient market hypothesis [1]. However, the mathematical apparatus of this theory (a model based on fractional Brownian motion) “did not keep up” with the substantial concept [2]. The lack of an adequate mathematical fractal dynamics pricing model at the time of the fractal market hypothesis formation prevented the formation of a full theory.

The attempts to revise the classical theory are due to the development peculiarities of market relations and the observable volatility of asset price dynamics under the influence of stylized facts of market participants [3], namely:

- excess volatility of asset returns, which can not be estimated by traditional methods of economic processes;
- heavy tails: the distribution of returns, indicating asymmetry of the market, which contributes to the growth of risks and the likelihood of extreme events;
- autocorrelation in returns, where homogeneous assets can exhibit the absence of dependence of profitability increments and the presence of a significant long-term memory of economic processes that can find manifestation in homogeneous processes of market relations
- volatility clustering, where the jumps in profitability tend to be followed by jumps of the opposite sign, significant for the market and asset price dynamics, which contribute to the probability of significant losses;
- volume/volatility correlation: trading volume is positively correlated with market volatility. Moreover, trading volume and volatility show the same type of “long-term memory” behavior.

The study of these phenomena began in the 1980s of the XX century [4]. However, mathematical modeling of individual stylized facts was first carried out by researchers at the beginning of the XXI century [5–7]. Currently, representatives of various scientific schools have shown that market development features are directly related to risk assessment and the need to use predictive mathematical models for adequate asset management solutions aimed at the stable manifestation of economic processes. It is important to note that a universal mathematical model of market asset price dynamics has not yet been found. For example, conducted within the European Central Bank in 2014 and based on the data analysis of the developed economies of the EU countries, the studies are aimed at finding a theoretical model that explains the market relations phenomena [8]. It is not possible to use the considered approach to predict the processes of emerging markets. Moreover, the study of cryptocurrency price

dynamics by the representatives of the European mathematical school in 2017 showed the features of forecasting the use of assets in cyberspace [9].

In this regard, the observation made in 2019 in the field of stochastic financial mathematics is interesting [10]. Analyzing the stylized facts of economic development on a large statistical material, the authors found that emerging markets behave similar to prediction markets, which confirms the role of general and specialized information in banking. An attempt to connect the stylized facts of market phenomena and the behavior of economic agents involves multi-agent models, including those with artificial intelligence, where market participants implement a relatively rational asset management strategy aimed at maintaining profit and risk management [11–13]. However, criticisms against multi-agent forecasting models, especially in emerging markets, remain valid [14].

In complex forecasting models of a highly volatile non-traditional market, the use of “non-standard” models is promising. Thus, the main asset pricing theorem was proved for markets where mathematical modeling was not possible [15].

In 2018, representatives of the University of Jerusalem introduced the notion of fully incomplete markets and attempted mathematical prediction of an asset hedging strategy [16]. To calculate asset price dynamics and manage financial risks, it is necessary to use comprehensive information about real prices and virtual derivative financial instruments.

The variety of methods and models used in modern financial mathematics shows that a unifying concept that generalizes the classical one and explains the stylized facts of market relations is not represented in modern science. The most systematic and consistent explanation of stylized facts of economic development is obtained within the concept of a fractal market, involving the dependence of the predicted value of asset price dynamics on

the history of market development. This article analyzes the concept.

MODELS BASED ON SELF-SIMILAR PROCESSES

The key assumption of the fractal market hypothesis is self-similarity of dynamic price series of assets. As a rule, the price dynamics of financial market assets is modeled using self-similar processes. This is supported by statistical observations and economic arguments [17].

Self-similarity is a consequence of a large number of market participants with different investment horizons and acting in the same conditions. Moreover, market participants behave in the same way with respect to their investment horizon, providing invariance of market characteristics relative to the time scale of asset use. The Hurst exponent H [17] is the statistical characteristic of scale invariance. Its value ranges from 0 to 1. For Brownian motion, underlying the classical models of the volatile market, the value of the Hurst exponent equals to 0.5. A value H in the range 0.5–1 indicates persistent (trend-stable) dynamics in the time series. A value in the range 0 – 0.5 indicates antipersistent dynamics in the time series and demonstrates the property of returning to the average value.

The mathematical apparatus to describe self-similar random processes was proposed by A.N. Kolmogorov. Methods for obtaining accurate numerical market predictions related to asset pricing have been developing for about half a century. However, no decisive results such as the Black-Scholes model have been received yet. The reason is that the use of fractional Brownian motion for asset price modeling in the stock market faces a difficult problem. Unlike classical mathematical modeling, models based on the fractional Brownian motion have arbitrage opportunities that cannot be described by the rational pricing theory.

For a long time, researchers believed that the existence of arbitrage opportunities was inextricably linked with autocorrelation and the memory of financial time series. A deeper penetration into the mathematics of the fractal market shows that arbitrage, autocorrelation, and self-similarity are due to various factors. Work [18] provides examples of Gaussian random processes that have the same long-term memory as the processes based on fractional Brownian motion with the Hurst exponent greater than 0.5, and at the same time lead to arbitrage-free market models. To build the price model, work [18] used the idea of a moving average, which successfully connects the mathematical apparatus with market realities understood by the financier.

Nevertheless, most researchers find it more promising to use precisely the fractional Brownian motion to build a market model. Replacing Ito integration by Wick integration can solve the availability problem of arbitrage opportunities [19, 20]. Experience has shown that, modified integration has still no convincing economic interpretation. Therefore, it is advisable to be careful when using mathematical modeling with Wick integration.

The solution to minimizing financial risks by mathematical modeling of pricing financial derivatives lies in complete accounting of the characteristics of trading financial instruments in a particular financial market. The fractal market with proportional transaction costs is arbitrage-free. The exact pricing financial derivatives in such a market is fundamentally impossible; it is only possible to establish more or less accurate price limits that do not allow arbitrage. However, the fractal market hypothesis attracts the participants by the opportunity to minimize financial risks of asset management.

Classical predictive models suggest that a random process with underlying Brownian motion describes the risky asset price dynamics. Namely, let $S(t)$ be the price of the risky asset

at time t . Then the return for time interval Δt is as follows:

$$\frac{S(t+\Delta t)-S(t)}{S(t)} = \mu\Delta t + \sigma\Delta W(t), \quad (1)$$

where $\mu + \frac{\sigma^2}{2}$ — is the expected return; σ — is

the return volatility; $\Delta W(t) = W(t+\Delta t) - W(t)$; $W(t)$ — is the so-called Wiener random process (Brownian motion). Value $\Delta W(t)$ is considered normally distributed with an average value of 0 and dispersion Δt . It is assumed that for different values of t , increments $\Delta W(t)$ are independent (unless the time intervals overlap).

Wiener processes belong to self-similar random processes. In general, a random market process is self-similar if a change in the time scale leads to a change in the spatial scale, and the probabilistic characteristics of the process remain unchanged. More precisely, random process $X(t)$, $t \geq 0$, is called self-similar if for any $a > 0$ we can find $b > 0$ so that random processes $X(at)$ and $bX(t)$ have the same probabilistic characteristics. If parameter b is also related to parameter a so that $b = a^H$ for some constant H for all $a > 0$, constant H is called the Hurst exponent and it is said that the process is self-similar with the Hurst exponent H . For the Wiener process, the Hurst exponent equals 0.5.

Levy processes are used in the models if the changes in return at disjoint time intervals are considered independent. Models based on Levy processes provide a good approximation of real price series, sometimes much better than classical models [21]. They allow considering such features of financial time series as asymmetry and heavy tails of probability distributions, and thereby more adequately assess risks (for example, ignoring heavy tails leads to underestimation of risks associated with extreme events). This is achieved by the fact that Levy processes are determined by a larger number of parameters than Wiener processes. Typically, four pa-

rameters are used. In some way, two of them are similar to the parameters of the Wiener process: μ is the shift parameter (similar to the average value that may be determined in the Levy process); σ is the scale parameter (similar to the average deviation that may not be determined the Levy process). The other two parameters allow considering the features of time series not captured by Wiener processes: β is the skewness parameter (allows considering the asymmetry in the differences between the probability distributions in the loss zone and in the zone of inflated expectations).

Work [21] showed that using Levy processes to describe the returns of world stock indices provides satisfactory results. At the same time, it is possible to consider the dynamic features of financial series, missing in classical models. Similar results are obtained regarding the Russian market [22].

Predictive ability is an important property of the model. To be considered qualitative and predictively valuable, the model should be sufficiently stable with respect to small fluctuations in the initial data and relatively small shifts along the time axis. In this regard, increasing number of parameters allows for a more accurate calibration on historical data, but the stability of the estimates is problematic. Data analysis shows that models with a normal distribution show good results for periods of 1–2 months. With a forecast period of more than 200 days, both classical models and those based on Levy processes are not entirely reliable. Finally, for periods of 100–150 days, models based on Levy processes provide the best result [23].

The use of non-classical models for the Russian market is more significant. For example, for the DJA index, the distributions in the corresponding Levy processes are close to normal, and both are consistent with empirical data. It is no longer the case for the RTS index due to high transaction costs (we also include the costs due to insufficient liquidity).

Fractional Brownian motion is a basic example of a self-similar random process with dependent increments. The dependence of increments makes it possible to simulate processes with long-term memory using fractional Brownian motion. The phenomena related to the trend formation are explained within the framework of such models.

The most systematic and consistent explanation of stylized facts of economic development is obtained within the concept of a fractal market, involving the dependence of the predicted value of asset price dynamics on the history of market development.

Applying financial time series models based on self-similar processes can face fundamental difficulties, regardless of the processes involved: with dependent or independent increments. In the classical Black-Scholes-Merton model, pricing is based on the fact that this model has an equivalent martingale probability measure for price stochastic processes. Substantially, this measure can be interpreted as some rational forecast, and the price of a derivative instrument is determined considering this forecast with respect to its future prices. In general, there is an infinite family of “rational forecasts” for self-similar random processes with independent increments. Accordingly, there appears an interval of prices interpreted as “fair”. Sometimes, but not always, it is possible to estimate the boundaries of these intervals. These boundaries are often shallow. In models using fractional Brownian motion, with the Hurst exponent other than 0.5, there is no “rational forecast” (equivalent martingale measure), and there are arbitrage opportunities. Building pricing models within such

models is only possible considering the features of the real financial market functioning. Transaction costs are among these features.

To manage financial risks in difficult market conditions when forecasting asset price dynamics, using the fractal modeling method is promising.

In classic models, the price of a derivative is determined by replicating strategies. In the presence of transaction costs, an exact replication may be too expensive. It is then replaced by a similar one, obtained as a result of solving the problem of stochastic control by dynamic programming methods. The solution to this problem in many cases is too complicated (even considering today's computing power). Facilitations are achieved by narrowing the class of acceptable investment strategies, for example, portfolio rebalancing can only be possible at fixed intervals. In this case, it is possible to obtain more or less acceptable estimates of the trading boundaries using upper and lower hedging [24]. Work [25] presents fundamentally important results of the estimates of the trading boundaries obtained under general assumptions. The authors managed to connect trading volumes, liquidity and dynamic parameters of price movement and to get estimates allowing for optimal trading strategies [26]. These works make relevant the issue of a consistent use of the so-called market time in models. Technically, this concept was used in many works. The results obtained in these works open up new possibilities for the Tobin tax. In our opinion, the studies indicate quite clearly that in financial market models it is advisable to link time with financial events, and not just with the rotation of the Earth around the Sun [25, 26].

To manage financial risks in difficult market conditions when forecasting asset price dynamics, using the fractal modeling method is promising.

FRACTIONAL BROWNIAN MOTION AND MARKET MODELS

Formally, fractional Brownian motion with the Hurst exponent H , $0 < H < 1$, is a stochastic process $\{B^H(t)\}$, where random variables $B^H(t)$ are normally distributed for all times t and $B^H(0) = 0$, the mean value of $B^H(t)$ is 0 for any t , and the covariance of $B^H(t)$ and $B^H(s)$ is as follows:

$$E[B^H(t)B^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \quad (2)$$

In an equivalent way, we can assume that the variance $B^H(t)$ is proportional to t^{2H} (in the case of the Wiener process, the dispersion is proportional to t).

The trajectory of fractional Brownian motion is a fractal object with a fractal dimension $D = 2 - H$.

By means of fractional Brownian motion, it is possible to build market models with many important properties, whose manifestation is demonstrated by real markets. We will call such models fractal markets for short.

One of the most important and studied is a model similar to the classical (1), where the risky asset price dynamics is described as follows:

$$\frac{S(t+\Delta t) - S(t)}{S(t)} = \mu\Delta t + \sigma\Delta B^H(t). \quad (3)$$

The behavior of the autocovariance yield function with the lag τ is similar to the behavior of the function $2H(2H-1)\tau^{2H-2}$ (we take a period equal to 1). For all values of the Hurst exponent, autocorrelation tends to 0 with an increase in the time lag.

At $H > 0.5$, autocorrelation is positive and decreases more slowly, the higher the value

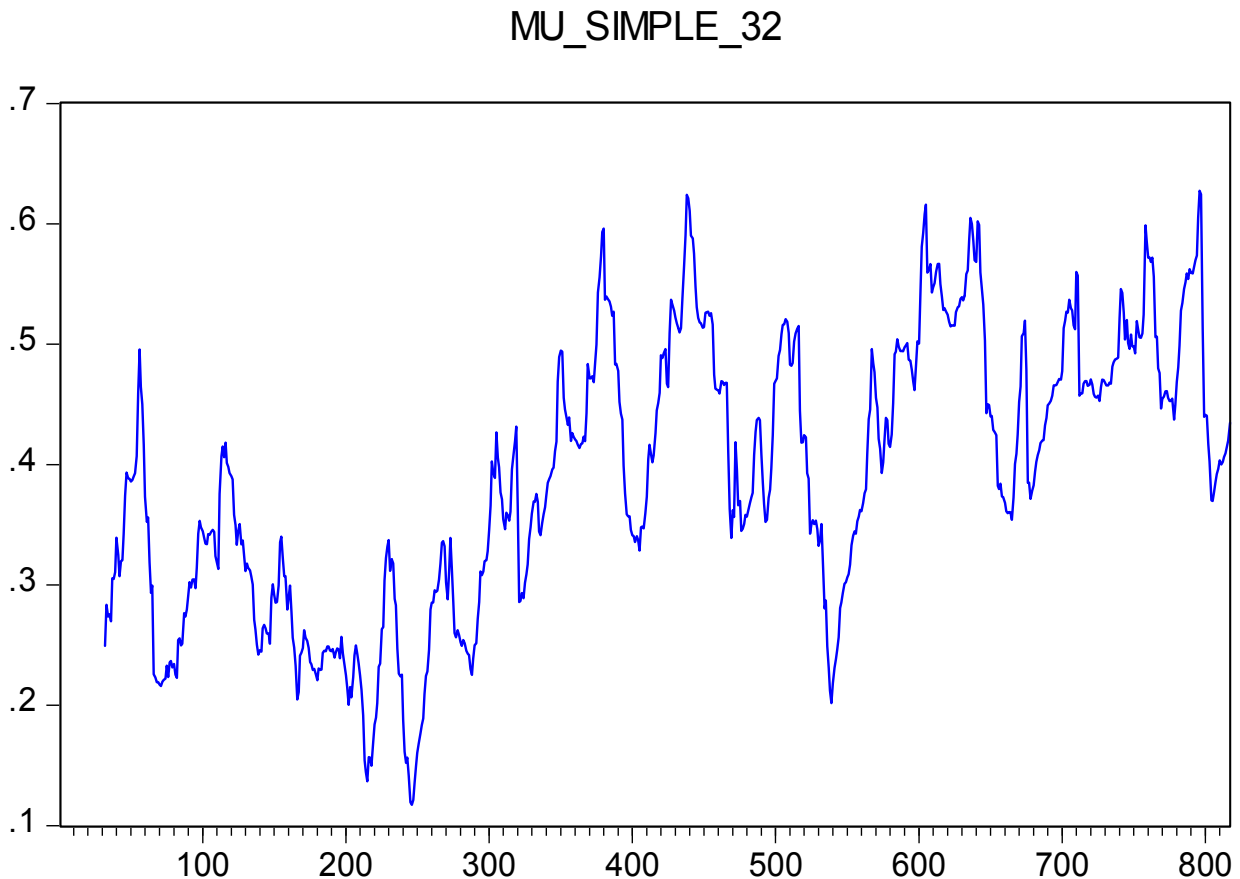


Fig. 1. Graph of $\mu(t)$

Source: compiled by the authors.

of H is. For example, at $H = 0.8$, autocorrelation remains quite noticeable (approximately 0.2) even at $\tau = 10$. This case corresponds to persistence.

At $H < 0.5$, autocorrelation becomes negative at $\tau < 1$, reaches its minimum value, and then tends to zero with increasing lag. This case corresponds to antipersistence.

These properties of the Hurst exponent are associated with crisis phenomena. Empirical observations allow us to conclude that a decrease in the fractal dimension of the price trajectory precedes large changes in the markets. The fractal characteristics of markets in the period up to 2014 were analyzed in work [27]. With this in mind, studying the dynamics of the Hurst exponent becomes relevant. This problem was studied in works [28, 29], as well as the concept of the fractality index

μ associated with the Hurst exponent by the relation $H \approx 1 - \mu$.

The fractality index dynamics allows a statistically reliable description and, due to this can be used for forecasting. Work [30] proposed promising econometric approaches to describing the dynamics of the Hurst exponent.

FRACTALITY INDEX

Values characterizing the fractal structure of the market are used to model volatility. The asset price is seen as a continuous stochastic process. The amplitude $A(\delta) = h(\delta) - l(\delta)$ is used as a measure of volatility over an interval of length δ , where $h(\delta)$ is the maximum and $l(\delta)$ is the minimum price in this interval.

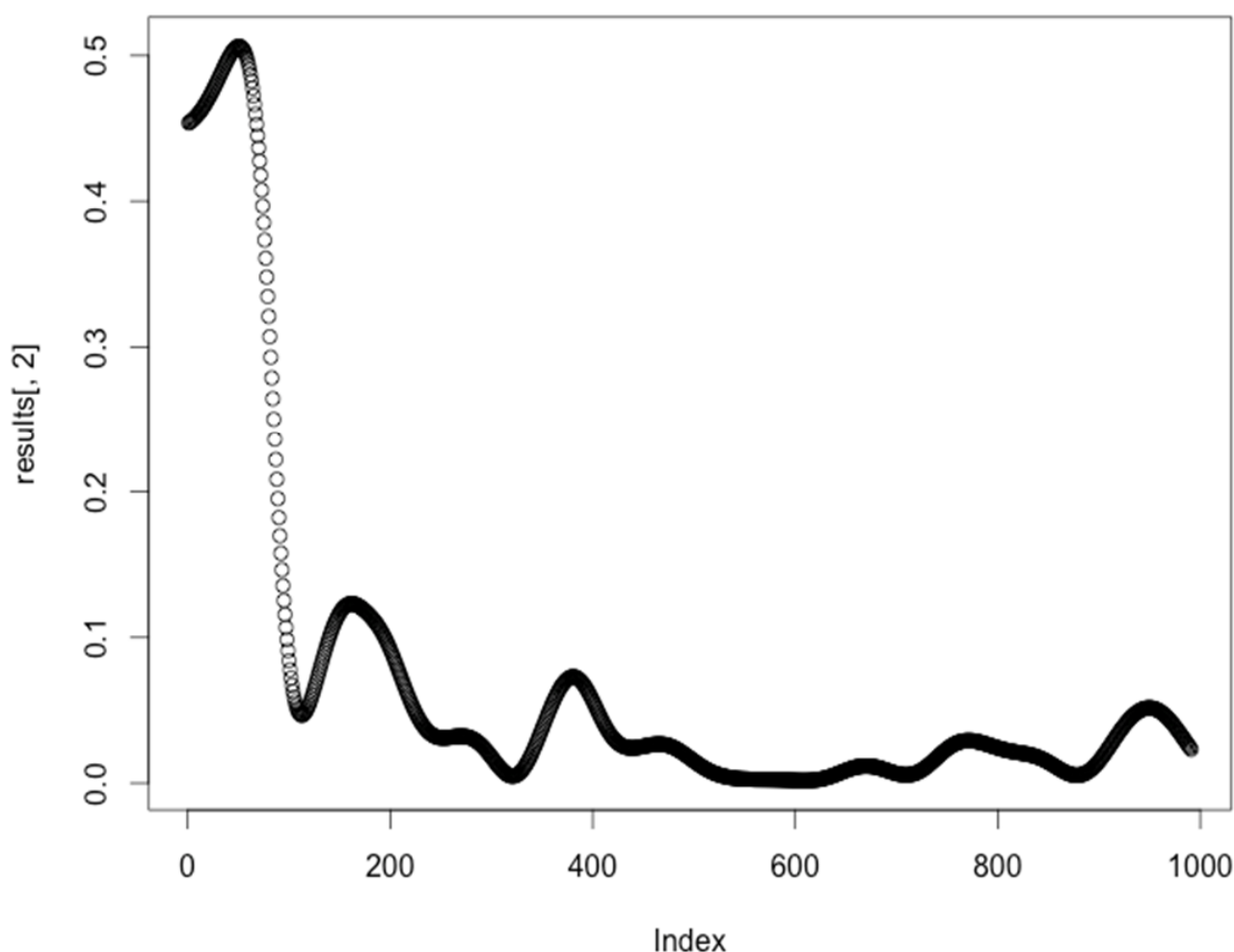


Fig. 2. Graph of $R^2(\omega)$.

Source: compiled by the authors.

Two values δ_0 and δ_c , are selected, with $\delta_c = n\delta_0$. Two values δ_c is usually called the characteristic scale. At time t , the interval $[t - \delta_c, t]$ is considered. Let δ be a divisor of a characteristic scale and a multiple of the minimum. The total of the amplitudes at these intervals is denoted by $V(\delta)$. The regression is considered

$$\log V(\delta) = \alpha - \mu \log(\delta). \quad (4)$$

In [28] it was shown that regression (4) has a very high coefficient of determination, that almost coincides with 1, in a rather wide range (the authors considered the ratio of the characteristic and the minimum scale from 8 to 1024). Thus, the estimate of μ is practically

independent of the choice of divisors, and we can consider the dynamic quantities $\mu(t, \delta_0, \delta_c)$ and $\alpha(t, \delta_0, \delta_c)$. As a rule, $\delta_0 = 1$, and dynamic quantities are denoted by $\mu_{\delta_c}(t)$ and $\alpha_{\delta_c}(t)$.

The function μ (unlike α) does not depend on the base of the logarithm in equality (4) and is an intrinsic characteristic of the fractal structure of the financial series. In work [28], the value μ is called the fractality index. When the minimum scale tends to 0, the fractality index tends to the value $D - 1$, where D is the fractal dimension of the price process. Moreover, the convergence turns out to be very fast (quick approach to asymptotics), which allows us to estimate the fractal dimension using very few observations.

As a consequence of the very high coefficient of determination of regression (4), we can use simplified estimates of the parameters μ_0 , α_0 (assuming $\delta_0 = 1$):

$$\begin{aligned}\mu_s &= \log_{\delta_c} V(\delta_0) - \log_{\delta_c} V(\delta_c); \\ \alpha_s &= \log_{\delta_c} V(\delta_0).\end{aligned}\quad (5)$$

which gives a decomposition of volatility on a characteristic scale:

$$\log_{\delta_c} V(\delta_c) = \alpha_s - \mu_s \approx \alpha - \mu. \quad (6)$$

USING FRACTAL CHARACTERISTICS TO FORECAST VOLATILITY

Work [29] considers regression models of volatility components α and μ that can be used to forecast future dynamics of the volatility of the foreign exchange market. While the future value is predicted in most models, but, as a rule, over a rather short interval, the fractal model allows predicting only the growth direction for α and μ , but over a sufficiently long interval (from one to eight months).

An empirical fact is used to build an econometric model: the function $\mu(t)$ has a quite clearly defined quasicyclic structure (Fig. 1). It should be noted that the quasicyclicity of fractal characteristics (in particular, the most dynamic series of Hurst) was noted and discussed at a qualitative level earlier [1, 2]. Since the quantity μ has a much faster asymptotics than the estimate of the R/S -analysis, it is natural to expect that the quasicyclicity of the function $\mu(t)$ is more pronounced.

Thus, it is logical to use periodic functions to model the fractality index:

$$\hat{\mu}(t) = \sum_{i=1}^k [a_i \sin(\omega_i t) + b_i \cos(\omega_i t)]. \quad (7)$$

The econometric model corresponding to (7) is constructed as follows. The equation is considered

$$\mu(t) = x + b_1 \sin(\omega t) + b_2 \cos(\omega t) + \varepsilon(t). \quad (8)$$

The frequency ω runs from 0 to 0.1 with 0.0001 per step. For each value of ω , the coefficient of determination of regression is determined (8). The resulting function $R^2(\omega)$ has pronounced extrema. It can be obtained for any segment of the time series $[T_0, T_1]$. A typical graph is shown in Fig. 2.

The smallest maximum with the highest determination coefficient is the main, trending frequency. Besides, there are frequencies of quasicycles — usually three or four.

Thus, both the extreme frequencies and the corresponding values of the determination coefficients turn out to be functions of two parameters: T_0 — the starting point and the length $\Delta = T_1 - T_0$ of the interval on which the econometric model (window width) is built. In this case, the picture shown in Fig. 2, at large intervals of the values of the parameters T_0 , Δ does not change qualitatively and changes little quantitatively, which confirms the quasicyclic nature of the structure. At the same time, at some values of T_0 , phase transitions take place. The main trend frequency in (8) bifurcates with the subsequent “overflow” — the attenuation of the initial “hump” and the growth of a new one.

These ideas were used in work [30] to forecast trends in the ruble exchange rate. At the same time, the regressions had a rather high coefficient of determination: $R^2 \sim 0.7 \div 0.75$. Backtesting the model showed that the direction prediction of the trend is correct in 60–70% of cases. The 2008 crisis found good agreement with the model.

Works [31, 32] presented data on the values of the Hurst exponent in the stock market, based on the analysis of voluminous statistical material. These data generally confirm the indicated pattern. In this regard,

observed in 2019, the increase in the values of the Hurst exponent in the domestic oil sector is alarming. Typical for the Russian stock market, the values of the Hurst exponent close to 0.6 (Aeroflot — 0.58–0.63; Gazprom — 0.53–0.60; Sberbank — 0.57–0.64; Rosneft — 0.53–0.57 in 2014–2018) were replaced in the first half of 2019 by the higher ones (Tatneft — 0.70; Surgutneftegas — 0.77; Rosneft — 0.72).

We refer to the study by a scientific school of the Utrecht University (Netherlands) that estimated the “normal” values of the Hurst exponent for various sectors: information technology — 0.50–0.67; finance — 0.38–0.62; raw materials sector — 0.38–0.63 [33].

The presented justification of the feasibility of using fractal models of asset price dynamics and their practical application in the financial sector can help minimize risks and strengthen the stable development of market relations.

There is no martingale measure in the fractal market. Therefore, there are arbitrage opportunities associated with the features of the Ito integral. Mathematically, the situation of forecasting price dynamics can be corrected by using Wick integration [19]. However, this integration method has not yet received an economic interpretation that is adequate to modern conditions of market relations. This approach can be easily explained by a discrete approximation of fractional Brownian motion, which serves as the main tool for calculations. A brief description of the discrete approximation follows next.

Let the time interval $[0; T]$ be divided into n equal intervals. For each n , we can calculate the coefficients $k_{l,i}^{(n)}$, $l=1, \dots, n$, $i=1, \dots, l$, so that the sums

$$B^H(t) \approx \sum_{i=1}^l k_{l,i}^{(n)} \xi_i, \quad (9)$$

where ξ_i — are random variables taking one of the two values $\{-1; 1\}$ approximate the values for $t = l \cdot \frac{T}{n}$ in the interval $[0; T]$.

Then,

$$\Delta B^H(t) = k_{l+1,l+1} \xi_{l+1} + \sum_{i=1}^l (k_{l+1,i} - k_{l,i}) \xi_i \quad (10)$$

[we omit the index n in the notation $k^{(n)}$ from (9)]. Equation (10) allows approximating the risky asset price in the fractal market at sufficiently large n .

Assuming that $\Delta t = \frac{T}{n}$ and $S_0 = S(0)$, we have:

$$S(\Delta t) = S_0 (1 + \mu \Delta t + k_{1,1} \xi_1); \quad (11)$$

$$S(2\Delta t) = S(\Delta t) \left(1 + \mu \Delta t + k_{2,2} \xi_2 + (k_{2,1} - k_{1,1}) \xi_1 \right) \dots \quad (12)$$

Ito integration corresponds to the usual multiplication of brackets. Wick integration corresponds to a multiplication where the terms containing ξ_i^2 , are discarded. For example, when calculating $S(2\Delta t)$ the term $k_{1,1}(k_{2,1} - k_{1,1})$, obtained by term-by-term multiplication of (11) and (12), at $\xi_1^2 = 1$, should be discarded. There is still no economically rational explanation why such terms should be discarded. We should be cautious about the results obtained by Wick integration.

CONCLUSIONS

Let us dwell on the results associated with pricing in markets with transaction costs. Studies [25, 34] suggested an approach to describe optimal strategies in markets with transaction costs.

Under general assumptions, the proportion of capital invested in the risk component should be within the boundaries

$$\pi_- = \frac{\rho - \lambda}{\gamma \sigma^2} \text{ and } \pi_+ = \frac{\rho + \lambda}{\gamma \sigma^2}, \quad (13)$$

where ρ — is the excess return; γ — is the relative risk aversion; ε — is the spread between supply and demand prices, and the quantity λ is the following

$$\lambda = \gamma \sigma^2 \left(\frac{3}{4\gamma} \pi_*^2 (1 - \pi_*)^2 \right)^{1/3} \varepsilon^{1/3} + O(\varepsilon) \quad (14)$$

$$\text{with } \pi_* = \frac{\rho}{\gamma \sigma^2}.$$

For example, the calculations according to formulas (13) and (14) for ordinary shares of Sberbank at the beginning of 2014 yielded the values $\pi_- = 45.6\%$, $\pi_+ = 48.2\%$. The liquidity premium calculated by the methodology of work [25] equaled 0.04%. For less attractive and liquid assets, the buy and sell limits were much lower, and the liquidity premium increased sharply. For example, it was 0.15% for JSCB Primorye.

A significant number of recent studies have been devoted to modeling volatility using fractional Brownian motion. It is possible to explain the effects of short-term and long-term memory, the paradox of the “volatility smile” and some other features in terms of the constructed models [35].

The concept of Rough Fractional Stochastic Volatility (RFSV) became widespread [36, 37]. The RFSV concept generalizes models used for

over 20 years (see [38]). In the standard stochastic volatility model described by equations

$$\frac{dS(t)}{dt} = \mu(t, S(t))dt + \sigma(t)dW^{(1)}(t); \quad (15)$$

$$d(\ln \sigma(t)) = k(\theta - \ln \sigma(t))dt + \gamma dW^{(2)}(t), \quad (16)$$

it is proposed to use the fractional Brownian motion instead of the Wiener process $W^{(2)}(t)$.

Research in this area was stimulated by the empirically revealed stable pattern: the volatility dynamics is fractal by nature, the Hurst exponent for process $W^{(2)}(t)$ equals 0.1 for fixed-income instruments. This Hurst exponent corresponds to a very high variability of volatility with a tendency to return to its average values. This observation can significantly improve volatility forecasts, and, most importantly, much more accurately than with other models, describe possible risks and implied volatility of asset price dynamics. The proposed approach is also promising in the formation of forecast models of asset price dynamics of derivative financial instruments [37]. Moreover, the fractal volatility parameters demonstrate predictive power regarding extreme events in the financial sector. An example is the collapse of Lehman Brothers and other US investment banks in 2008, which caused the global financial and economic crisis [36].

The presented justification of the feasibility of using fractal models of asset price dynamics and their practical application in the financial sector can help minimize risks and strengthen the stable development of market relations.

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Authors' declared contribution:

Yarygina I. Z. — disclosed the features of the economic content of value of bank assets and developed recommendations for assessing financial risks based on mathematical methods for forecasting economic processes.

Gisin V. B. — analyzed the features of fractal Brownian motion to describe price dynamics; described the method for assessing the boundaries of the fair prices of financial assets in the fractal market.

Putko B. A. — used the observed quasicyclicity of the fractality index to construct an econometric model with a fractality index as an explanatory variable and periodic harmonics as explanatory variables. A long-term forecast was based on this model. The result of backtesting the model was given.

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